MATH 136-01

Chapter 5 Topic Review Sheet

October 13, 2004

This is a list of terminology and topics covered in the fifth chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text.

We covered all sections, 5.1 – 5.4.

Measuring Distance Traveled. (Section 5.1) When velocity is constant, the formula $distance = velocity \cdot time$ allows us to compute distance travelled, given time. In this section, we use this formula to estimate distance, when velocity is not constant. Given velocity $v(t)$ measured every Δt seconds, we obtain upper and lower estimates on total distance traveled. We first plot the velocity data, connecting the points with a smooth curve. We estimate the area under this curve by adding up the area of rectangles of width Δt, and of height equal to the velocity measured at either the start of the interval (in a **left-hand sum**), or at the end of the interval (in a **right-hand sum**). Averaging the two estimates gives the best overall estimate. Moreover, if $v(t)$ is either increasing or decreasing on an interval $a \leq t \leq b$, the error involved in our left-hand or right-hand estimate is at most $|v(b) - v(a)| \cdot \Delta t$.

The Definite Integral (Section 5.2) In this section, we define the **definite integral** $\int_a^b f(x)\,dx$, where $f(x)$ is continuous on $[a,b]$. The function $f(x)$ is the **integrand**, and a and b are the **limits (or bounds) of integration.** The definite integral is defined as the limit of the left-hand or the right-hand **Riemann sums**, as the number of subintervals n goes to infinity. These sums are computed using the values of $f(x)$ at the points $a = x_0, x_1, \ldots, x_{n-1}, x_n = b$ over n subintervals of equal length whose union is $[a,b]$. To get from one endpoint to the next, we add the increment $\Delta x = \frac{b-a}{n}$.

Using a specified number of subintervals for $[a,b]$, we may best estimate $\int_a^b f(x)\,dx$ by computing a left-hand Riemann sum, and a right-hand Riemann sum, and averaging the two. (This amounts to computing the sum using the **Trapezoid rule.**) We note that if $f(x)$ is decreasing on $[a,b]$, then the left-hand sum is an overestimate, while the right-hand sum is an underestimate. If $f(x)$ is increasing on $[a,b]$, these roles are reversed.

When $f(x)$ is positive on $[a,b]$, we interpret $\int_a^b f(x)\,dx$ as the area under the graph of f between a and b. If $f(x)$ is negative on $[a,b]$, then $\int_a^b f(x)\,dx$ is the *negative* of the area between the graph of f and the x-axis, between a and b. If $f(x)$ changes sign on $[a,b]$, then areas above the x-axis contribute positively, and areas below the x-axis contribute negatively to $\int_a^b f(x)\,dx$.

Interpretations of the Definite Integral. (Section 5.3) In this section, we discuss **total change** and **average value** of a function, in terms of definite integrals.

First, the total change in $F(t)$ between $t = a$ and $t = b$ is given by $\int_a^b F'(t)\,dt$. This relates to the Fundamental Theorem of Calculus, which is discussed in Section 5.4.

Second, the average value of $f(x)$ from $x = a$ to $x = b$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. In terms of the graph of $f(x)$, rearranging gives (average value of f on $[a,b]$) $\cdot\,(b-a) = \int_a^b f(x)\,dx$, so the average value of f is the height of a rectangle whose width is $b - a$ and whose area is $\int_a^b f(x)\,dx$. The average value is the precise value k for which the function spends half the time below k and half the time above k.

Theorems about Definite Integrals. (Section 5.4) This section begins with the **Fundamental Theorem of Calculus**, which states that if f is continuous on $[a,b]$ and $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt \;=\; F(b) - F(a).$$

This provides a way of computing definite integrals exactly, by finding an appropriate $F(t)$. Such a function is called an **antiderivative** of $f(t)$. Theorems 5.2, 5.3, and 5.4 discuss **properties of the definite integral** which you should know, such as the fact that if c is a number in the interval $[a,b]$, and f is continuous on $[a,b]$, then we can split the integral like this: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. In addition to the properties listed in these theorems, we also have facts about definite integrals when the integrand has even or odd symmetry. Specifically, if f is even, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ for any a, and if g is odd, then $\int_{-a}^a g(x)\,dx = 0$ for any a. All of these properties can be proven using the definition of the definite integral as a limit of Riemann sums.