MATH 136-01

Chapter 4 Topic Review Sheet

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This is a list of terminology and topics covered in the fourth chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text.

The focus of the chapter is on applications of the first and second derivative, in particular, finding extrema of functions. This is useful for curve sketching and for solving "real-world" problems. We will primarily cover sections 4.1, 4.3 and 4.5.

Using First and Second Derivatives (Section 4.1) This section builds on Section 2.4, which made the connection between the sign of f' and the rate of change of f ($f' > 0$ implies f is increasing and $f' < 0$ implies that f is decreasing), and Section 2.6, which made the connection between the sign of f'' and the concavity of f ($f'' > 0$ implies f is concave up and $f'' < 0$ implies that f is concave down). In this section, **local minima** and **local maxima** are introduced. We will say **local extrema** when we want to refer to either local minima or maxima. According to Theorem 4.1, if f is differentiable in an open interval containing a local extrema p, then $f'(p) = 0$. Points p where $f'(p) = 0$ or $f'(p)$ is undefined are called **critical points** of the function (the value $f(p)$ at a critical point is called the **critical value**). Thus we may begin our search for local extrema of a differentiable function by locating the critical points of f. Since not all critical points are local extrema it is necessary to test each critical point to determine the behavior of the function at the point.

By the **first derivative test**, a critical point p of a continuous function f is a local minimum if f' changes from negative to positive at p and a local maximum if f' changes from positive to negative at p. It is often useful to draw a "first-derivative number line" to record the sign of the derivative in different intervals.

By the **second derivative test**, if f is a twice differentiable function and $f'(p) = 0$, p is a local minimum if $f''(p) > 0$ (f is concave up at p) and a local maximum if $f''(p) < 0$ (f is concave down at p).

A point p is called an **inflection point** of f if the concavity of f changes at p. This will occur either if f' has a local minimum or maximum at p or if f'' changes sign at p. In this case, the "second-derivative number line" is useful for sketching graphs and determining whether a point with vanishing second derivative is actually an inflection point.

Optimization (Section 4.3) This section introduces **global extrema**. Here we are concerned with the behavior of a function on an interval, which might be a bounded interval or the entire real line. A point p is a **global minimum** if $f(p) \leq f(x)$ for all other points x in the interval and a **global maximum** if $f(p) \geq f(x)$ for all other points x in the interval. If the interval is closed (that is, it includes its endpoints), then the global extrema will be located at the critical points of f on the interval or at the endpoints. If the interval is open or is the entire real line, then the global extrema (if they exist) will occur at the critical points of f on the interval. However, if f takes on values larger than any critical value on the interval as x approaches the endpoints of the interval or $\pm\infty$, then f will have no global maximum on the interval. The analogous statement holds for global minima. In certain situations, it will be sufficient to determine whether or not a function is **bounded** on an interval and to find a **lower bound** or an **upper bound** for the function on the interval.

Optimization and Modeling (Section 4.5) This section applies the ideas of Section 4.3 to "real world" problems. (The problems in this section are also often called "word problems.") In these problems, we are asked to find an extreme value of some quantity, often of a physical nature. The problem is presented to us in a prose description, which usually contains numerical information and might also be accompanied by a diagram illustrating the physical setup. In order to apply the techniques of Section 4.3, we must identify the important quantities in the problem, represent them by variables, determine the relationship(s) between the variables, and represent these relationships as functions. The resulting equation(s) is a **mathematical model** of the problem. Once we have the model, we apply the techniques of Section 4.3 to find the global extrema of the function. Finally, we have to interpret these extrema in the context of the original problem.

Theorems about Continuous and Differentiable Functions (Section 4.7) In practice, we assume that the calculations we use to produce solutions to extrema problems will lead to solutions of the problem. In fact, this is not automatic, but depends on the properties of continuous functions and differentiable functions. The most important of these are:

The Extreme Value Theorem A continuous function on a closed interval has a global maximum and a global minimum on the interval.

The Mean Value Theorem A differentiable function has the property that between any two points a and b, there is a third point c, so that the slope of the tangent line at c is the same as the slope of the secant line between $(a, f(a))$ and $(b, f(b))$.

The Mean Value Theorem is a consequence of the Extreme Value Theorem. In turn, the Mean Value Theorem is used to demonstrate the following:

The Increasing Function Theorem If a function has strictly positive derivative on an interval then it is increasing on the interval. If the derivative is only non-negative, then the function is non-decreasing.

The Constant Function Theorem If a differentiable function has derivative equal to 0 everywhere in a closed interval, the function is constant.