MATH 136-01

Chapter 3 Topic Review Sheet

September 15, 2004

This is a list of terminology and topics covered in the third chapter of *Calculus*, D. Hughes-Hallet, et al. 3rd edition. Please consult the text for full definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. The primary focus of Chapter 3 is on the differentiation "rules" such as the product, quotient and chain rules. You should be comfortable applying the appropriate rule or rules to find the derivative of a given function.

Powers and Polynomials (Section 3.1) Differentiation is a linear operation. In other words, the **derivative of a constant multiple** of a function is the constant times the derivative of the function and the **derivative of a sum (difference)** of two functions is the sum (difference, respectively) of the derivatives. In shorthand notation, $[cf(x)]' = cf'(x)$ and $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$. Both of these rules can be proven using the definition of the derivative and the properties of limits on p. 65. Another important differentiation rule is the **power rule**, $[x^n]' = nx^{n-1}$. Together, these rules allow us to easily differentiate polynomials.

The Exponential Function (Section 3.2) What distinguishes an exponential function from other functions is that its derivative is a constant multiple of itself. In particular, the exponential function $f(x) = e^x$ is its own derivative, that is, $[e^x]' = e^x$. The base e is precisely the special constant for which this is true. For bases other than e, we have the formula $[a^x]' = \ln(a)a^x$.

The Product and Quotient Rules (Section 3.3) In this section the **product rule** and **quotient rule** are derived. The product rule states that the derivative of a product of two functions is the derivative of the first times the second plus the first times the derivative of the second. In shorthand notation, $[fg]' = f'g + fg'$. The quotient rule states that the derivative of a quotient of two functions is the bottom times the derivative of the top minus the top times the derivative of the bottom, all divided by the bottom squared. In shorthand, $[f/g]' = (f'g - fg')/g^2$. The quotient rule can be derived using the product rule.

The Chain Rule (Section 3.4) The **chain rule** is used to differentiate the composition of two functions. This is particularly useful since many functions can be written in the form $f(g(x))$. The chain rule is $[f(g(x))]' = f'(g(x)) \cdot g'(x)$. This is often referred to as "the derivative of the outside times the derivative of the inside" but be aware of the fact that the derivative of the outside is *being evaluated* at the inside function. If y is a function of u (say $y = f(u)$) and u is a function of x (say $u = g(x)$), then y is also a function of x ($y = f(g(x))$) and the chain rule is often written as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The Trigonometric Functions (Section 3.5) Two important derivative formulas are $[\sin x]' = \cos x$ and $[\cos x]' = -\sin x$. Note that these formulas are for x in radians and can be derived from the important limits

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{\cos h - 1}{h} = 0.$$

Using the quotient rule, we can then derive the derivatives of the other trigonometric functions. For example, $[\tan x]' = \sec^2 x$.

Applications of the Chain Rule (Section 3.6) The chain rule can be used to derive several differentiation formulas for the inverses of functions whose derivatives we already know. For example, we can find the derivative of $\ln x$ by using the equation $e^{\ln x} = x$, the chain rule and the derivative of e^x. This gives $[\ln x]' = 1/x$. We can also derive the formulas

$$[a^x]' = (\ln a)a^x, \quad [\arctan x]' = \frac{1}{1+x^2} \quad \text{and} \quad [\arcsin x]' = \frac{1}{\sqrt{1-x^2}}.$$

Implicit Functions (Section 3.7) Sometimes a function is not *explicitly* given in the form $y = f(x)$ but rather defined **implicitly** such as $y^5 - xy = -6\sin x$. In this case, we can still calculate the derivative of y, dy/dx, by treating y as a function of x and using the chain rule. This process is called **implicit differentiation**.

Linear Approximation and the Derivative (Section 3.9) If the function $f(x)$ is differentiable at $x = a$, then the **tangent line approximation** is $y = f(a) + f'(a)(x-a)$. This is the best linear approximation to f at $x = a$. The expression $f(a) + f'(a)(x-a)$ is the **local linearization** of f near $x = a$ and the error $E(x)$ is given by $E(x) = f(x) - f(a) - f'(a)(x-a)$. If f is differentiable at $x = a$, then

$$\lim_{x \to a} \frac{E(x)}{x-a} = 0.$$

Using Local Linearity to Find Limits (Section 3.10) When finding the limit of a quotient for which both the numerator and denominator are heading to zero, we may apply **L'Hopital's rule** to calculate the limit provided that the functions in question are differentiable. Specifically, if f and g are differentiable at $x = a$ and $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Other versions of L'Hopital's rule exist for limits involving $\pm\infty$. These are summarized in the box at the bottom of p. 156.