MATH 136-01

Chapter 2 Topic Review Sheet

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This is a list of terminology and topics covered in the second chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for full definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. The primary focus of Chapter 2 is the concept of the **derivative**. Before investigating the differentiation "rules" presented in Chapter 3, you should have a firm grasp of the definition of the derivative and the qualitative information about a function (increasing, decreasing, concavity) revealed by its first and second derivatives. You should also be *comfortable* sketching the graph of the derivative $f'(x)$ given the graph of $f(x)$.

How Do We Measure Speed? (Section 2.1) The **average velocity** of an object over an interval is the net change in the object's position during the interval divided by the change in time. The object's position at time t is denoted $s(t)$, and its velocity at time t is denoted by $v(t)$. Geometrically, the average velocity over an interval $a \leq t \leq b$ is the slope of the segment joining the points $(a, s(a))$ and $(b, s(b))$ on the graph of $s(t)$. The **instantaneous velocity** of an object at time $t = a$ is found by taking the limit of its average velocity over shorter and shorter time intervals. Thus we can say that the instantaneous velocity gives the **slope of the curve** $s(t)$ at a point.

Limits (Section 2.2) In this section, we define the **limit** of a function $f(x)$ as x approaches c, denoted $\lim_{x \to c} f(x)$. There is an informal, intuitive definition given on p. 63. The precise "ϵ-δ" definition is given on p. 64. It is particularly important that you be familiar with the properties of limits in the table on p. 65. **One-sided limits** are instances where the function has a limit as x approaches c from the right, or from the left, but not both. **Limits at infinity** deal with the "end behavior" of a function $f(x)$, that is the behavior of $f(x)$ as x approaches infinity or negative infinity.

The Derivative at a Point (Section 2.3) This section generalizes the concepts from Section 2.1 to arbitrary functions. A **difference quotient** is used to define the **average rate of change** of a function $f(x)$ on the interval $[a, a + h]$. Taking the limit as $h \to 0$ of the average rate of change yields the **instantaneous rate of change** of $f(x)$ at $x = a$, or the **derivative of f at a,** denoted $f'(a)$. The function $f(x)$ is said to be **differentiable** at $x = a$ if this limit exists. We may visualize the derivative $f'(a)$ as the slope of the curve $f(x)$ at the point $(a, f(a))$. We may approximate the derivative at a point by using smaller and smaller values of h to estimate the limit which defines $f'(a)$. We may also derive $f'(a)$ directly from the limit definition, using the properties of limits from p. 65.

The Derivative Function (Section 2.4) Varying the point at which we compute the derivative yields the **derivative function** $f'(x)$. We say f is **differentiable** at each x-value where $f'(x)$ is defined (i.e. the corresponding limit exists). Some of the functions we encounter are **differentiable everywhere**, i.e. at every point in their domains. We examine the derivative function in several ways: Graphically, f is **increasing** whenever $f' > 0$ and **decreasing** whenever $f' < 0$. You should be comfortable with giving a rough sketch of the function $f'(x)$, given the graph of $f(x)$. Numerically, f' can be approximated at a point, as was done in Section 2.3. Algebraically, given a formula for $f(x)$ we can often derive a formula for $f'(x)$. This is done in Section 2.4 for **constant functions** and **linear functions**. These and other examples are done to illustrate the **power rule** for taking derivatives. (More differentiation rules will follow in Chapter 3.)

Interpretations of the Derivative (Section 2.5) If $y = f(x)$, instead of denoting the derivative function by $f'(x)$, we may write $\frac{dy}{dx}$. This **Leibniz notation** suggests the interpretation of the derivative as a quotient of an "infinitesimal" change in y over an infinitesimal change in x. Although not formally correct, this notion is useful in explaining how the units on $\frac{dy}{dx}$ are given by (the units of y)/(the units of x). Examples in this section show how to interpret the derivative in practical situations, in terms of its value and its units.

The Second Derivative (Section 2.6) The **second derivative** of a function $f(x)$, denoted $f''(x)$ or $\frac{d^2y}{dx^2}$, is defined to be the derivative of the derivative of f. The sign of f'' tells us whether f' is increasing or decreasing (in other words, whether the slopes of the tangent lines are increasing or decreasing). It therefore tells us about the concavity of f: f is **concave up** whenever $f'' > 0$ and **concave down** whenever $f'' < 0$. You should be comfortable with giving a rough sketch of the function $f''(x)$, given the graph of $f(x)$ or $f'(x)$. If $s(t)$ denotes the position of an object at time t and $v(t)$ denotes its velocity, then $s''(t) = v'(t)$ is the **acceleration** of the object at time t. This is also denoted by $a(t)$.

Continuity and Differentiability (Section 2.7) We say a function f is **continuous** at $x = c$ if $\lim_{x \to c} = f(c)$. Note that this says three things: f is defined at $x = c$, the limit of $f(x)$ *exists* at that x-value, and that the function value and limit value coincide. A function failing any one of these three at $x = c$ will have a **discontinuity** at that point. We say f is **continuous on an interval** $[a, b]$ if f is continuous at every point in the interval. You should be familiar with the "pencil and paper" version of continuity: f is continuous on $[a, b]$ if you can draw the whole graph of f on $[a, b]$ *without lifting your pen from the paper.* Of particular importance are the properties of continuity as given in the two theorems on p. 95. Notice that if $f(x)$ is differentiable at a point $x = a$, then $f(x)$ is necessarily continuous at $x = a$ (Theorem 2.4), but that it doesn't work the other way around. The **absolute value function** provides an excellent example that *continuity does not imply differentiability.* You should be able to recognize points where a function $f(x)$ is discontinuous or non-differentiable from its graph. For instance, $f(x)$ fails to be differentiable at a point if the graph has a discontinuity, or a sharp corner, or a vertical tangent line at that point.