MATH 136-01

Chapter 11 Topic Review Sheet

December 7, 2004

This is a list of terminology and topics covered in the eleventh chapter of *Calculus*, D. Hughes-Hallet, et al. 3rd edition. Please consult the text for full definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. This chapter focuses on **differential equations**. We covered sections 11.1 – 11.5.

Differential Equations (Section 11.1) An **ordinary differential equation (ODE)** is an equation that relates the derivative(s) of an unknown function to the function itself. A **first-order** ODE involves the first derivative of the unknown function, but no higher derivatives. A **second-order** ODE involves the second derivative of the unknown function, but no higher derivatives. If the ODE involves $\frac{dy}{dx}$ for instance, then a **solution** is a FUNCTION $y(x)$ which satisfies the ODE. To check that a given function is a solution to a given ODE we PLUG IT IN to both sides of the differential equation.

Solving an ODE will result in the **general solution**, which is a family of functions involving one or more arbitrary constants. (For instance, a second-order ODE will involve two constants, while a first-order ODE will involve only one.) Given **initial condition(s)**, we can solve for the values of these arbitrary constants, thus determining a **particular solution**. An **initial-value problem** is an ODE, together with initial condition(s).

Slope Fields (Section 11.2) We can visualize a first-order ODE $\frac{dy}{dx} = f(x, y)$ using a **slope field**. Any solution $y(x)$ has the property that its slope at any point (x_0, y_0) will be given by $\frac{dy}{dx}|_{(x_0,y_0)} = f(x_0, y_0)$. At a selection of points, we plot a small arrow with its tail at the selected point, and with its head indicating the direction of the tangent line to the solution curve passing through that point. In the resulting **slope field**, solution curves may be sketched in by "following the arrows." That is, starting at any point, tracing a curve which keeps the arrows tangent to its path will yield a solution curve. Sometimes the shape of these curves will suggest the general solution (e.g. circles would mean solutions are of the form $x^2 + y^2 = r^2$). Therefore a slope field, together with "guess and check," may be used to determine solutions in certain simple cases.

Euler's Method (Section 11.3) Slope fields suggest a simple numerical method of approximating the solution to a differential equation. One such method is **Euler's Method**, which essentially steps along the slope field by a fixed increment Δx. Given an initial starting point $y(x_0) = y_0$ and an ODE $\frac{dy}{dx} = f(x, y)$, we compute recursively the approximation to the solution using the formula

$$x_{n+1} = x_n + \Delta x \tag{1}$$
$$y_{n+1} = y_n + f(x_n, y_n)\Delta x \tag{2}$$

starting with (x_0, y_0). Thus if we began with $x_0 = 1$ and set $\Delta x = 0.2$, we would need to make 5 steps to approximate the solution at $x = 2$. The formula for y_{n+1} is derived by the fact that $\frac{\Delta y}{\Delta x}$ is the slope at the previous point x_n, y_n which in turn is given by plugging into the function $f(x, y)$ on the right-hand side of the differential equation.

The error in Euler's method is directly proportional to Δx. To improve the approximation we let Δx approach 0, but this requires more steps of Euler's method be computed.

Separation of Variables (Section 11.4) Differential equations of the form $\frac{dy}{dx} = g(x)h(y)$ can be solved explicitly using the method of **separation of variables**. Rewrite the equation with all terms involving y on the left-hand side, and all terms involving x on the right-hand side. Then integrate both sides, and solve for y in terms of x. An example of an ODE which can be solved this way is $\frac{dy}{dx} = ky$, where k is a constant. We separate variables, and then solve $\int \frac{dy}{y} = \int k\,dx$. The general solution is $y = ce^{kx}$, where c is an arbitrary constant. Similarly, the equation $\frac{dy}{dx} = k(y - A)$, where A is a constant, has the general solution $y = ce^{kx} + A$.

Growth and Decay (Section 11.5) Differential equations may be used to model real-world processes, such as exponential growth of an investment (from the compounding of interest), or the temperature of an object over time (according to Newton's Law of Heating and Cooling). One common model is when the rate of change of a quantity is proportional to the quantity itself (eg. population) leading to the ODE $\frac{dP}{dt} = kP$. The solutions to this ODE are given by $P(t) = P_0e^{kt}$ where P_0 is the initial amount. When $k > 0$, this represents growth, while $k < 0$ represents decay. Similarly, if the temperature of an object is given by a function $T(t)$ satisfying **Newton's Law of Cooling or Heating**, then the rate of change in the temperature of the object is proportional to the difference in temperature between the object and its surrounding medium. This results in the equation $\frac{dT}{dt} = k(T - A)$, where A is the temperature of the surrounding medium (assumed constant.) Solutions are given by $T(t) = ce^{kt} + A$.

Families of solutions which are exponential often involve an **equilibrium solution** which is constant for all values of the independent variable. The corresponding solution curve is a horizontal line. This equilibrium is **stable** if nearby solutions approach it as the independent variable approaches infinity, or **unstable** if nearby solutions veer away from it as the independent variable approaches infinity. The type of equilibrium point can be determined from a slope field or by examing the behavior of the general solution as the independent variable approaches ∞.