MATH 136-01
Chapter 10 Topic Review Sheet

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This is a list of terminology and topics covered in the tenth chapter of *Calculus*, D. Hughes-Hallet, *et. al.* 3rd edition. Please consult the text for definitions, statements of properties, and numerous examples and exercises. Terms in bold face are defined in the text. We covered sections 10.1, 10.2, 10.3. Throughout this chapter we will be working with functions that can be differentiated arbitrarily many times.

Taylor Polynomials. (Section 10.1) The Taylor polynomials for a function $y = f(x)$ near a point $x = a$ are polynomials used to approximate f near a. We have already seen one instance of this, the equation of the tangent line for a differentiable function f at a, which is the **first Taylor polynomial** at the point. Here we will denote the tangent line by $P_1(x)$. The formula for P_1 is

$$P_1(x) = f(a) + f'(a)(x - a).$$

Notice, the tangent line is the unique line passing through $(a, f(a))$ with slope $f'(a)$. (Keep in mind that x is the variable and a is a fixed value.)

The **Taylor polynomial of degree** n for f near a is denoted $P_n(x)$. It is given by the formula

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Using sigma notation, this formula is written as

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

Here $f^{(k)}(a)$ denotes the k^{th} derivative of f at a and $f^0(a) = f(a)$. Note that $0! = 1$ by definition. Taylor polynomials are characterized by the following property: For $k = 0, \ldots n$, the k^{th} derivative of P_n at a is equal to the k^{th} derivative of f at a. In symbols, $P_n^{(k)}(a) = f^{(k)}(a)$. Intuitively, Taylor polynomials give good approximations to f near a, that is, for x near a, $P_n(x) \approx f(x)$. The further x is from a, the worse the approximation. The higher the degree of the Taylor polynomial, the better the approximation.

Taylor Series (Section 10.2) The Taylor polynomial of degree n for f near a, $P_n(x)$, is the n^{th} partial sum of a power series. This power series is the **Taylor series** for f at a. Essentially, we are letting $n \to \infty$ in the expression for $P_n(x)$.

In sigma notation, we have

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots.$$

Note that the Taylor series is an example of a power series from Section 9.4. It is important to remember that this series has an interval of convergence! For any x for which the series converges, $f(x)$ is equal to the series expansion evaluated at x.

For example, we derived the following important Taylor series in class:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - + \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - + \cdots
\end{aligned}$$

All three series converge for all $x \in \mathbb{R}$. If we plug in $x = 1$ into the Taylor series for e^x we obtain an infinite series for the number e:

$$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Note that the series for $\sin x$ contains only odd powers so that $\sin(-x) = -\sin x$, that is, $\sin x$ is an odd function. Similarly, $\cos(-x) = \cos x$. The Taylor series for $\cos x$ is easily derivable from the Taylor series for $\sin x$ by differentiating both sides. These series are related by Euler's famous formula:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

Finding and Using Taylor Series (Section 10.3) This section shows how to construct new Taylor series from known series by **substitution**, **differentiation** and **integration**. By substitution, we can, for example, produce the Taylor series for e^{2x} from the series for e^x. That is, since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

by replacing x by $2x$ we find that

$$\begin{aligned}
e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots \\
&= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots
\end{aligned}$$

Since differentiation of series amounts to term-by-term differentiation, we can calculate the Taylor series for the derivative of a function by differentiating its series. Similarly, since integration of series amounts to term-by-term antidifferentiation, we can calculate the Taylor series for the antiderivative of a function by integrating its series. For example, using the Taylor series for $1/(1+x^2)$ (geometric series with ratio $r = -x^2$)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - + \cdots$$

we can integrate both sides to obtain the Taylor series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - + \cdots$$

These techniques give quick and simple methods for computing Taylor series that are far easier than doing repeated differentiation. Finally, in this section there are several examples that use the Taylor polynomial of a function to draw conclusions about the behavior of the function near the point $x = a$.