MATH 132, Spring 2006

Computer Lab #1

Numerical Integration

DUE DATE: Tuesday, Feb. 28, in class.

The goal for this lab project is to use Maple to compute, compare and visualize different numerical techniques for approximating definite integrals. The algorithms we have been studying are left-hand and right-hand sums, the midpoint rule, the trapezoid rule and Simpson's rule. In this lab you will compute the errors of each method and determine the rate of improvement as the number of subdivisions is increased. Ultimately, you will identify which numerical integration technique is the most accurate and by how much. This lab is intended to explore and supplement the material in Sections 7.5 and 7.6 of the course text.

It is **required** that you work in a group of two or three people. Any help you receive from a source other than your lab partner(s) should be acknowledged in your report. For example, a textbook, web site, another student, etc. should all be appropriately referenced. Please turn in one report per group, listing the names of the groups members at the top of your report. Be sure to answer all questions carefully and neatly, writing in complete sentences. You should type your answers, leaving space for mathematical calculations or graphs. You may also include calculations and graphs in an appendix at the end of your report. There are only **two graphs** to be turned in for this report.

Getting Started: Useful Maple Integration Commands

To execute some of the commands needed for this lab, you must load three packages which come with the current version of Maple. Typing

```
with(student):  with(plots):  with(Student[Calculus1]):
```

will load the pertinent packages. These packages contain commands for doing numerical integration as well as the animation routines you will be running. Don't forget to load these packages **every time** you work on the lab!

Recall that left-hand and right-hand sums, the midpoint rule, the trapezoid rule and Simpson's rule are methods for approximating the signed area under the graph of $f(x)$ over an interval $[a, b]$. We first divide the interval $[a, b]$ into n equal pieces, (called a partition), yielding n subintervals of width $\Delta x = (b - a)/n$. Then, we construct rectangles (or trapezoids) over each subinterval by choosing a representative height determined by the function. For a left-hand sum, we evaluate the function at the left endpoints of each subinterval, while for a right-hand sum, we use the right endpoints. For the midpoint rule we evaluate the function at the midpoint of each subinterval, while the trapezoid rule averages the left and right-hand sums. (Recall from class that this is identical to finding the area under the trapezoid between the endpoints — hence the name!) Simpson's rule is a weighted estimate of the midpoint rule (2/3) and the trapezoid rule (1/3).

In this lab, we will investigate the accuracy of a given numerical integration technique and compare methods to each other, drawing some important conclusions. For the lab, define the error of an approximation to be

$$\text{Error} = \text{Actual value} - \text{Approximation}$$

We begin with a left-hand sum. Typing

```
leftbox(x^2,x=1..2,12);
```

will draw 12 rectangles for a left-hand sum of $f(x) = x^2$ over $1 \leq x \leq 2$. Try it. Now draw 50 rectangles for the same function and same interval. Typing

```
rightbox(x^2,x=1..2,10);
```

draws the right-hand sum over the same interval but this time with 10 subintervals. To graph the midpoint rule, type

```
middlebox(x^2,x=1..2,12);
```

Notice that in this case, part of each rectangle is above the curve and part is below, leading to a better approximation than left-hand or right-hand sums.

It would be nice to compare the left-hand, right-hand and midpoint sums visually by placing them side by side each other. This is easily done using the `display` and `array` commands. To display the three sums described above side by side, type the following:

```
Lplot:=leftbox(x^2,x=1..2,12,title="Left-hand Sum"):
Rplot:=rightbox(x^2,x=1..2,12,title="Right-hand Sum"):
Mplot:=middlebox(x^2,x=1..2,12,title="Midpoint Rule"):
display(array([Lplot,Rplot,Mplot]));
```

The `title` command lets you know which graph is which. Notice how your plot reveals that the left-hand sum is an underestimate, the right-hand sum is an overestimate and the midpoint rule looks pretty good.

To evaluate these sums using one of our numerical techniques, we can use the `leftsum`, `rightsum`, `middlesum`, `trapezoid` and `simpson` commands. For example, typing

```
evalf(leftsum(x^2,x=1..2,12));
```

gives the value of the left-hand sum for $f(x) = x^2$ over $1 \leq x \leq 2$ with 12 rectangles (subdivisions). If you want more rectangles (and thus a more accurate answer), replace 12 with a larger value. The `evalf` command (standing for "evaluate using floating-point arithmetic") is necessary to obtain a numerical answer. For calculating right-hand sums use the command `rightsum` with exactly the same syntax. The other commands are used in a similar fashion.

Finally, we can numerically calculate a definite integral using the `int` command. This can also be used to do symbolic integration as well. Maple has its own built in numerical and symbolic integrators for obtaining answers. For example, to find the value of

$$\int_1^2 x^2 \, dx$$

you can type `int(x^2,x=1..2);` If you do not obtain a numerical expression after using the `int` command, then try using an `evalf` command to tell Maple that you want a numerical answer outputted. For instance,

```
evalf(int(sin(x^2),x=0..Pi));
```

gives a numerical approximation to the integral of $\sin(x^2)$ from $x = 0$ to $x = \pi$. Without the `evalf` command, you would not obtain a numerical value. To symbolically calculate the integral of x^2 you simply type `int(x^2,x)`.

Exercises:

1. The following questions involve the function $f(x) = 4 - x^2$. Before executing the numerical calculations, type `Digits:=20;` which will output a greater number of significant digits in all of your computations. Recall that we define the error of an approximation to be

$$\text{Error} = \text{Actual value} - \text{Approximation}$$

   (a) Plot a graph of the left-hand sum, right-hand sum and midpoint rule side by side for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ with $n = 12$ subdivisions. Turn in your graph. You may need to use the mouse to stretch the graph out to make a nicer figure.

   (b) Use the Fundamental Theorem of Calculus to compute the exact area under f over the interval $0 \le x \le 2$. You may check your answer with Maple.

   (c) Use Maple to calculate the left-hand and right-hand sums for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$ and 2000 subdivisions. At each stage compute the error with the value obtained in part **(b)**. You can easily use Maple to compute the error. Make a table of your values and the corresponding errors.

   (d) Use Maple to calculate the midpoint and trapezoid rules for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$ and 2000 subdivisions. At each stage compute the error with the value obtained in part **(b)**. You can easily use Maple to compute the error. Make a table of your values and the corresponding errors.

   (e) Use Maple to calculate Simpson's rule for $f(x) = 4 - x^2$ over the interval $0 \le x \le 2$ using $n = 2, 20, 200$ and 2000 subdivisions. What do you notice? What is the error? Experiment by increasing the number of digits precision to 50 and 100.

   (f) For the left-hand, right-hand, midpoint and trapezoid rules, discuss the improvement in each method as the number of subintervals increases. In other words, for each method, increasing n by a factor of 10, decreases the error by a factor of how much? How many decimal places of accuracy are gained when n is increased by a factor of 10?

   (g) Compare the error of the midpoint rule and trapezoid rule. Which method is more accurate and by what factor? Your answer motivates the definition of Simpson's rule.

2. The following questions involve the integral

$$\int_{-1}^{1} \sin(2x^2 + 1)\, dx$$

   (a) Use the `int` and `evalf` commands to numerically approximate the integral using Maple. Note that this integral is impossible to do other than through numerical techniques, so this answer is the best we can hope for.

   (b) Use Maple to calculate the left-hand and right-hand sums for this integral using $n = 2, 20, 200$ and 2000 subdivisions. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

(c) Use Maple to calculate the midpoint and trapezoid rules for this integral using $n = 2, 20, 200$ and 2000 subdivisions. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

(d) Use Maple to calculate the integral with Simpson's rule using $n = 2, 20, 200, 2000$ subdivisions. At each stage compute the error with the value obtained in part **(a)**. Make a table of your values and the corresponding errors.

(e) For all five numerical techniques, discuss the improvement in each method as the number of subintervals increases. In other words, for each method, increasing n by a factor of 10, decreases the error by a factor of how much? How many decimal places of accuracy are gained when n is increased by a factor of 10? Which method is the most effective?

(f) Compare the error of the midpoint rule and trapezoid rule. Which method is more accurate and by what factor? Are there any differences with your answer to **1(g)**?

3. One fun and informative feature of the student Calculus1 package is the ability to animate Riemann Sums. As the number of rectangles increases, the area under the curve is better approximated. The following command is an example of how to visualize this improvement for $f(x) = e^{-x^2}$ on the interval $1 \le x \le 2$ for 6 different partitions. The first partition you see should have 4 rectangles and should be a right-hand sum. Carefully type

```
ApproximateInt(exp(-x^2),1..2, output = animation, iterations = 6,
method = right, partition = 4, subpartition = all, refinement = halve);
```

To run the animation, use the mouse to click on the plot. On the toolbar at the top of the screen should appear a DVD-like panel from which you can play the animation. The double arrow commands control the speed of the animation (you may wish to slow it down). The arrow with a vertical bar next to it will play one frame at a time. Notice how the area estimate improves with each iteration.

How many subdivisions are there at each iteration of the animation? (ie. what are the n-values in each case?) Print out and turn in the plot which is obtained after running the animation through once (this should be the graph with the best approximation).

4. The velocity of a spaceship in km/hr is given by

$$v(t) = 104.2\, t^3\, e^{-0.15\, t^2}$$

where $t \ge 0$ is measured in hours. Use Maple to calculate the distance the ship travels in the first 12 hours of its journey. How far does the ship travel in the next 12 hours? Explain how you obtained your answers.