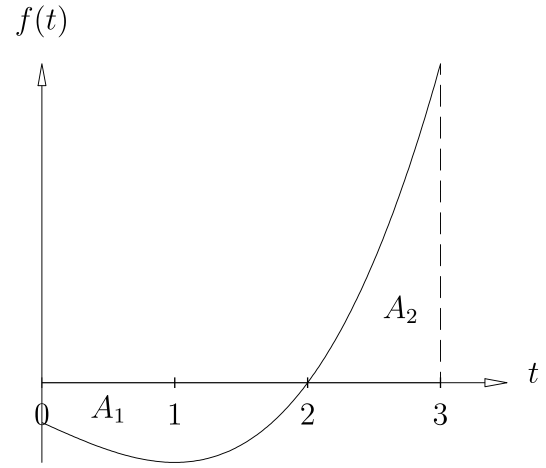
Math 132: Calculus for the Physical & Life Sciences 2

Spring 2006

Practice Final Exam Solutions

1. The graph of the function f is shown below. The areas shown are $A_1 = 6$ and $A_2 = 6.75$. Let F be the antiderivative of f that satisfies $F(2) = 3$.



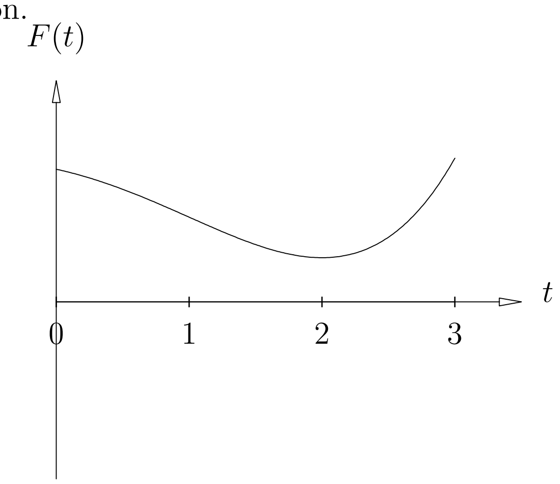
   (a) Compute $F(0)$ and $F(3)$.

   Solution. By the Fundamental Theorem of Calculus,

$$F(2) - F(0) = \int_0^2 f(x)\,dx = -6 \quad \text{and} \quad F(3) - F(2) = \int_2^3 f(x)\,dx = 6.75$$

   so $F(0) = F(2) + 6 = 9$ and $F(3) = F(2) + 6.75 = 9.75$.

   (b) On the axes below, sketch the graph of F. Be sure to label and maxima, minima, and points of inflection.



2. Compute the following integrals. Be sure to state clearly what method of integration you are using. If you use an entry from the table of integrals, state which one.

   (a) $\int 2x\cos(x^2)\,dx$

   Solution. Substitute $w = x^2$, $dw = 2x\,dx$:

$$\int \cos w\,dw = \sin w + C = \sin(x^2) + C$$

(b) $\int x^2 \cos(2x)\,dx$

Integrate by parts, with $u = x^2$, $v' = \cos(2x)$: $u' = 2x$ and $v = \frac{1}{2}\sin(2x)$, so

$$\int x^2 \cos(2x)\,dx = \frac{1}{2}x^2 \sin(2x) - \int x \sin(2x)\,dx$$

Integrate by parts again with $u = x$, $v' = \sin(2x)$: $u' = 1$ and $v = -\frac{1}{2}\cos(2x)$, so we get

$$\frac{1}{2}x^2 \sin(2x) - \left[-\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx\right]$$

Computing the last integral and simplifying gives

$$\frac{1}{2}x^2 \sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C.$$

(c) $\int \frac{36}{x(x-3)(x+3)}\,dx$

Solution. Use partial fractions. First write

$$\frac{36}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}$$

Clear denominators:

$$36 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3)$$

Plugging in $x = 0$ gives $36 = -9A$, so $A = -4$. At $x = 3$ we get $36 = 18B$, so $B = 2$ and at $x = -3$ we get $36 = 18C$, so $C = 2$. Thus the integral becomes

$$\int \frac{-4}{x}\,dx + \int \frac{2}{x-3}\,dx + \int \frac{2}{x+3}\,dx = -4\ln|x| + 2\ln|x-3| + 2\ln|x+3| + C$$

(d) $\int \frac{1}{(4-x^2)^{3/2}}\,dx$

Solution. Use trig substitution: $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $4 - x^2 = 4\cos^2\theta$, so

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{2\cos\theta\,d\theta}{(4\cos^2\theta)^{3/2}} = \int \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$

Now since $x = 2\sin\theta$ we have $\sin\theta = \frac{1}{2}x$ and $\cos\theta = \frac{1}{2}\sqrt{4-x^2}$, so $\tan\theta = \frac{x}{\sqrt{4-x^2}}$, so the integral equals $\frac{x}{4\sqrt{4-x^2}} + C$.

3. Determine whether the following improper integrals converge or diverge. Compute the value of the integral if it converges.

(a) $\int_3^6 \frac{1}{\sqrt{x-3}}\,dx$

Solution. The integral is improper at $x = 3$, so it becomes

$$\lim_{a\to 3^+} \int_a^6 \frac{1}{\sqrt{x-3}}\,dx = \lim_{a\to 3^+} 2\sqrt{x-3}\Big|_a^6 = \lim_{a\to 3^+} 2\sqrt{3} - 2\sqrt{a-3} = 2\sqrt{3}$$

so the integral converges.

(b) $\displaystyle\int_1^\infty \frac{1}{10000}\,dx$

Solution.

$$\lim_{b\to\infty}\int_1^b \frac{1}{10000}\,dx = \lim_{b\to\infty}\frac{x}{10000}\bigg|_1^b = \lim_{b\to\infty}\frac{b-1}{10000} = \infty,$$

so the integral diverges.

4. (a) Use the trapezoid rule with $n = 2$ subintervals to estimate $\displaystyle\int_1^5 \ln x\,dx$.

Solution. $\Delta x = (5-1)/2 = 2$, so the Trapezoid Rule gives an approximation of

$$\frac{1}{2}(\ln(1)+\ln(3))\cdot 2 + \frac{1}{2}(\ln(3)+\ln(5))\cdot 2 = 2\ln(3)+\ln(5) \approx 3.8$$

(b) Is the estimate in part (a) an overestimate or an underestimate? Explain.

Solution. The graph of $\ln x$ is concave down, so the Trapezoid rule gives an underestimate.

5. Let R be the region bounded by $y = e^{-x}$, $y = 0$, $x = 0$ and $x = 1$.

(a) Find the area of R.

Solution.

$$\int_0^1 e^{-x}\,dx = -e^{-x}\bigg|_0^1 = 1 - e^{-1}$$

(b) Find the volume of the solid obtained by revolving R about the axis $y = -2$.

Solution.

$$\int_0^1 \pi(e^{-x}+2)^2 - \pi(2)^2\,dx = \pi\int_0^1 e^{-2x} + 4e^{-x}\,dx = \frac{9}{2} - \frac{1}{2}e^{-2} - 4e^{-1}$$

6. Suppose $P(t) = \frac{1}{9}t^2$ is the cumulative distribution function for the length (in inches) of a certain population of worm, for $0 \le t \le 3$.

(a) Find the median length.

Solution. Solving $P(T) = \frac{1}{2}$ gives $T = \sqrt{9/2}$.

(b) Find the proportion of the population with length at least 2 inches.

Solution. $P(2) = \frac{4}{9}$ is the proportion of the population with length at most 2 inches, so the proportion with length at least 2 inches is $1 - \frac{4}{9} = \frac{5}{9}$.

(c) Find the density $p(x)$.

Solution. $p(x) = P'(x) = \frac{2}{9}x$.

(d) Find the mean length.

Solution. $\int_0^3 xp(x)\,dx = \int_0^3 \frac{2}{9}x^2\,dx = \frac{2}{27}x^3\bigg|_0^3 = 2.$

7. Find the arc length of the parametric curve $x = t^2$, $y = \frac{1}{3}t^3 + t$, $0 \le t \le 4$.

Ignore this question.

8. Both of the following series converge. One is geometric, and the other comes from evaluating a well known Taylor series. Identify which is which, and find their sums.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!} = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$

   Solution. This is the Taylor series for $\cos x$ evaluated at $x = \pi$, so it converges to $\cos \pi = -1$.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^k}{4^k} = 1 - \frac{\pi}{4} + \frac{\pi^2}{16} - \frac{\pi^3}{64} + \cdots$

   Solution. This is geometric with $a = 1$ and ratio $\frac{\pi}{4}$, so it converges to $\frac{1}{1-\frac{\pi}{4}} = \frac{4}{4-\pi}$.

9. Use the indicated method to determine whether or not the series converges. You do not need to compute the sums.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1+\sqrt{n}}{n-1}$; comparison test

   Solution. $\frac{1+\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{n^{1/2}}$. Since $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = 1/2 < 1$, it diverges. So by the Comparison Test, the series $\sum_{n=2}^{\infty} \frac{1+\sqrt{n}}{n-1}$ diverges.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$; integral test

   Solution. Since

$$\int_1^{\infty} \frac{x^2}{x^3+5}\,dx = \lim_{b\to\infty} \frac{1}{3}\ln(x^3+5)\Big|_1^b = \lim_{b\to\infty} \frac{1}{3}\ln(b^3+5) - \frac{1}{3}\ln(6) = \infty,$$

   the series diverges.

10. Use the ratio test to find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 3^n}$$

   Be sure to determine whether or not the series converges at each endpoint.

   Solution. Since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-1|n^2}{3(n+1)^2} = \frac{1}{3}|x-1|$$

   the ratio test implies that the series converges when $\frac{1}{3}|x-1| < 1$, which leads to the interval $-2 < x < 4$.

   At $x = -2$ the series becomes $\sum \frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. At $x = 4$, the series is $\sum \frac{1}{n^2}$ which converges since it is a p-series with $p = 2 > 1$. Therefore the interval of convergence is $-2 \le x \le 4$.

11. Let $f(x) = x^4 + x^3 + x^2 + x + 1$.

   (a) Find the third degree Taylor polynomial of f at $a = 1$.

   Solution. $f(1) = 5$, $f'(x) = 4x^3+3x^2+2x+1$ so $f'(1) = 10$, $f''(x) = 12x^2+6x+2$ so $f''(1) = 20$, and $f'''(x) = 24x+6$ so $f'''(1) = 30$. Therefore $P_3(x) = 5 + 10(x-1) + 10(x-1)^2 + 5(x-1)^3$.

(b) Use the error bound for Taylor polynomials to give an estimate of the maximum possible error at $x = 1.1$.

Solution. $f''''(x) = 24$, so we may use $M = 24$ in the error estimate. Thus $|E_3(1.1)| \leq \frac{24(1.1-1)^4}{4!} = 0.0001$.

(c) Compute both $f(1.1)$ and $P_3(1.1)$. What is the actual error?

Solution. $f(1.1) = 6.1051$ and $P_3(x) = 6.105$ so the actual error is 0.0001.

12. Use the Taylor series for e^x at $a = 0$ to find the Taylor series for $\dfrac{e^{x^2}-1}{x}$ at $a = 0$. Write the series using summation notation, and write out the first four nonzero terms.

Solution. Plugging x^2 into

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

gives

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$$

so subtracting 1 and dividing by x gives

$$\frac{e^{x^2}-1}{x} = x + \frac{x^3}{2!} + \frac{x^5}{3!} + \frac{x^7}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{n!}$$

13. Use two steps of Euler's method, with $\Delta t = 0.5$ to approximate the solution of the differential equation

$$\frac{dy}{dt} = 1 + ty$$

with initial condition $y(0) = 1$.

Solution.

n	t_n	y_n
0	0	1
1	0.5	1.5
2	1	2.375

Here's how the entries in the last column are computed: $y_1 = y_0 + f(t_0, y_0)\Delta x = 1 + f(0,1)(0.5) = 1 + 1(0.5) = 1.5$ and $y_2 = y_1 + f(t_1, y_1)\Delta x = 1.5 + f(0.5, 1.5)(0.5) = 1.5 + (1.75)(0.5) = 2.375$.

14. (a) Solve the separable equation $\dfrac{dy}{dt} = y^2t^2$.

Solution.

$$\int \frac{1}{y^2}\,dy = \int t^2\,dt \quad \Longrightarrow \quad -y^{-1} = \frac{1}{3}t^3 + C \quad \Longrightarrow \quad y = \frac{-1}{\frac{1}{3}t^3 + C}$$

(b) Find the solution which satisfies the initial condition $y(0) = 3$.

Solution. The initial condition implies $3 = -1/C$ so $C = -1/3$ and thus

$$y = \frac{-1}{\frac{1}{3}t^3 - \frac{1}{3}} = \frac{3}{1 - t^2}$$

15. Suppose a population of dandelions in a yard is being harvested in such a way that it obeys the differential equation

$$\frac{dP}{dt} = 0.2P - 300$$

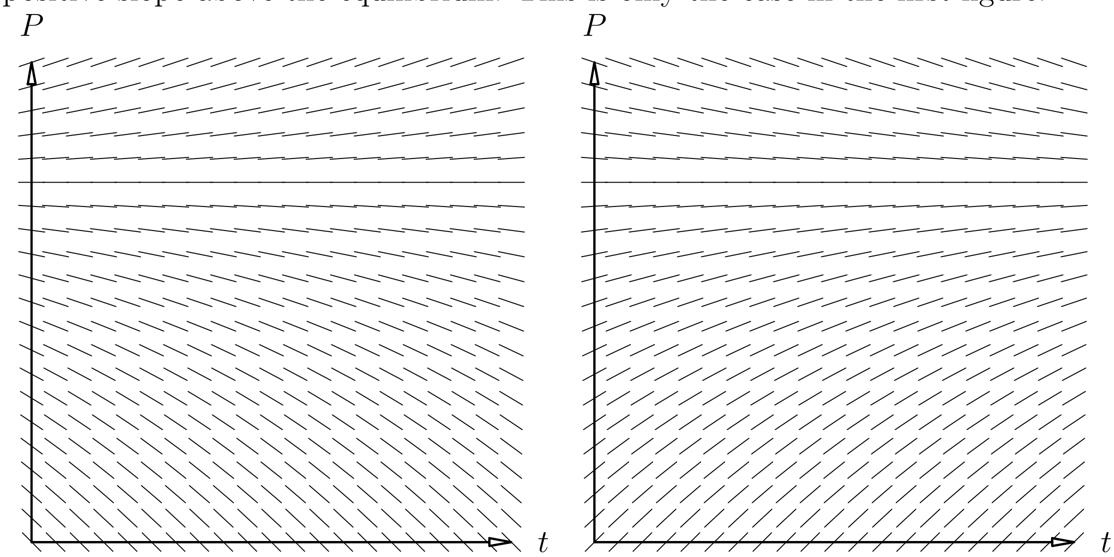
where t is measured in weeks.

(a) What is the equilibrium?

Solution. The right hand side of the equation is zero when $P = 1500$.

(b) Identify the figure below which represents the slope field for the differential equation. Label the equilibrium.

Solution. If $P > 1500$, then $0.2P - 300 > 0$ so the slope field lines should have positive slope above the equilibrium. This is only the case in the first figure.



(c) Is the equilibrium stable or unstable?

Solution. The equilibrium is unstable, since solutions move away from it as t increases.

(d) Suppose there are initially 1200 dandelions. Will the population of dandelions:

i. $\boxed{\text{be harvested into extinction}}$?

ii. grow without bound?

iii. level off at the equilibrium?

(Circle one answer.)

Solution. Since 1200 is less than the equilibrium of 1500, it is clear from the figure that the solution will eventually reach $y = 0$ (i.e. extinction).

16. When water drains from a tub, the rate at which the water level decreases is proportional to the square root of the water level. (This is called Torricelli's Law.)

(a) Write down a differential equation for the water level $y(t)$. Be sure to specify the sign of any constants in the equation.

Solution. $\frac{dy}{dt} = k\sqrt{y}$ for some constant $k < 0$.

(b) Solve this separable equation for y.

Solution. Separating variables gives

$$\int y^{-1/2}\, dy = \int k\, dt \quad \Longrightarrow \quad 2y^{1/2} = kt + C \quad \Longrightarrow \quad y = \frac{1}{4}(kt + C)^2$$

(c) Suppose the water level in a tub is initially 9 inches, and 5 minutes later the water level is 4 inches. How long will it take for the tub to completely drain?

Solution. $y(0) = 9$ implies $\frac{1}{4}C^2 = 9$, so $C = 6$. $y(5) = 4$ implies $\frac{1}{4}(5k + 6)^2 = 4$, so $k = -2/5$. Therefore $y(t) = \frac{1}{4}(-\frac{2}{5}t + 6)^2$, so $y(t) = 0$ when $t = 15$ minutes.

17. A savings account earns 2% interest compounded continuously. Suppose withdrawals are made at a constant rate of \$1000 per year.

(a) Write down a differential equation for the account balance $B(t)$.

Solution. $\frac{dB}{dt} = 0.02B - 1000$.

(b) Solve this equation for $B(t)$.

Solution. Writing the right hand side as $0.02(B-50000)$ and separating variables gives

$$\int \frac{dB}{B - 50000} = \int 0.02\, dt \quad \Longrightarrow \quad \ln(B - 50000) = 0.02t + C$$

So $B = 50000 + Ce^{0.02t}$.

(c) If the account initially contains \$10000, when will all the money in the account be depleted? What if the account initially contains \$20000?

Solution. $B(0) = 10000$ implies $10000 = 50000 + C$, so $C = -40000$. Thus $B(t) = 50000 - 40000e^{0.02t}$. Setting $B(t) = 0$ and solving gives $t = 50\ln(5/4) \approx 11.16$ years.

Setting $B(0) = 20000$ gives $C = -30000$, and setting $B(t) = 0$ gives $t = 50\ln(5/3) \approx 25.54$ years.

(d) For which initial amounts will the account never be depleted?

Solution. The original equation $\frac{dB}{dt} = 0.02B - 1000$ has equilibrium $B = 50000$. When $B > 50000$, $0.02B - 1000 > 0$, so for any initial data $B(0) > 50000$, the balance will increase for all time.

18. Shown below is the slope field for the differential equation

$$\frac{dy}{dt} = t - y^2$$

In the figure, both t and y vary from -1 to 1.

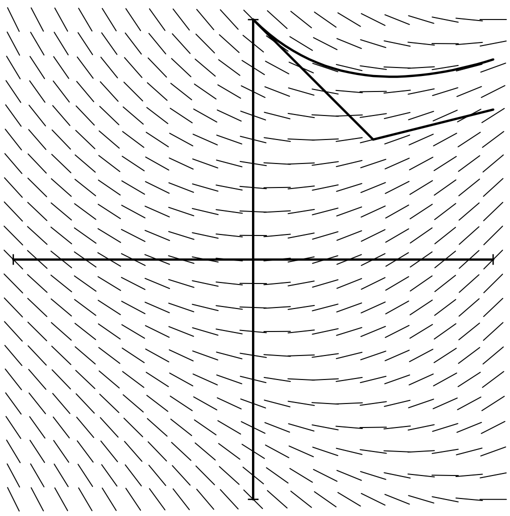
(a) On the axes provided, roughly sketch the solution of the differential equation with $y(0) = 1$.

(b) Compute two steps of Euler's method with $y(0) = 1$ and $\Delta t = 0.5$. Sketch the resulting approximate solution on the axes above. At $t = 1$, does Euler's method give an underestimate or an overestimate?

Solution.

$$y_1 = y_0 + f(t_0, y_0)\Delta t = 1 + f(0,1) \cdot 0.5 = 1 + (-1)(0.5) = 0.5$$
$$y_2 = y_1 + f(t_1, y_1)\Delta t = 0.5 + f(0.5, 0.5) \cdot 0.5 = 0.5 + (0.25)(0.5) = 0.625$$

Euler's method appears to give an underestimate at $t = 1$.



19. Solve each of the following separable equations.

(a) $\dfrac{dy}{dx} = e^{x-y}$

Solution. Writing $e^{x-y} = \frac{e^x}{e^y}$ and separating variables gives

$$\int e^y\,dy = \int e^x\,dx \quad \Longrightarrow \quad e^y = e^x + C \quad \Longrightarrow \quad y = \ln(e^x + C)$$

(b) $\dfrac{dy}{dt} = \dfrac{\cos t}{y+1}$, $y(0) = 2$

Solution.

$$\int y + 1\,dy = \int \cos t\,dt \quad \Longrightarrow \quad \frac{1}{2}y^2 + y = \sin t + C$$

Setting $y(0) = 2$ gives $C = 4$, so $\frac{1}{2}y^2 + y = \sin t + 4$.

(c) $\dfrac{dy}{dx} = \dfrac{3y}{x}$, $y(1) = 4$. Multiply by 2: $y^2 + 2y = 2\sin t + 8$. Now complete the square: $y^2 + 2y + 1 = 2\sin t + 9$. So $(y+1)^2 = 2\sin t + 9$, and thus $y = \sqrt{2\sin t + 9} - 1$. (The positive square root is used because of the initial condition $y(0) = 2$.)

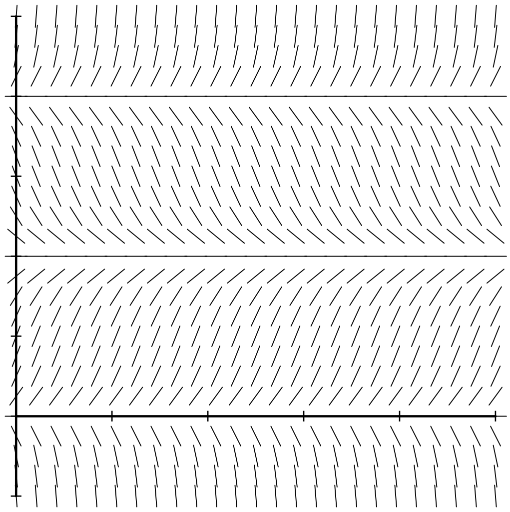
20. For the differential equation

$$\frac{dP}{dt} = P(P-2)(P-4),$$

(a) find the equilibria.

Solution. Solving $P(P-2)(P-4) = 0$ gives $P = 0, P = 2, P = 4$.

(b) roughly sketch the slope field.



(c) classify each equilibrium as stable or unstable.

Solution. The equilibrium $P = 2$ is stable, but the equilibria $P = 0$ and $P = 4$ are unstable.