Math 132: Calculus for the Physical & Life Sciences 2

Spring 2006

Solutions to Practice Questions for Midterm 3

1. Suppose $f(0) = -0.5$, $f'(0) = -1$ and $f''(0) = 2$.

   (a) Write down the Taylor polynomial of degree 2 for f near $a = 0$.

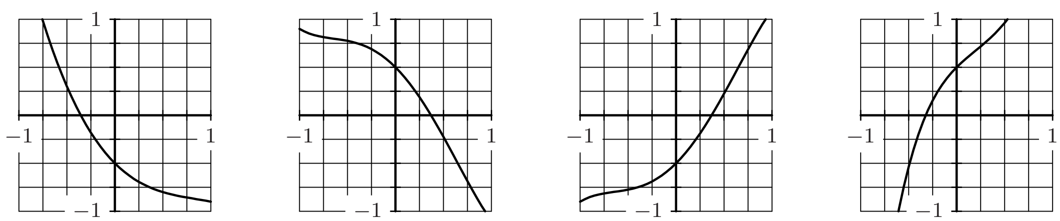
   Solution. $P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = -0.5 - x + x^2$.

   (b) Use your answer to part (a) to estimate $f(0.3)$.

   Solution. Plug 0.3 into P_2: $P_2(0.3) = -0.5 - (0.3) + (0.3)^2 = -0.71$.

   (c) Which (if any) of the following could be the graph of f? (Recall that $f(0) = -0.5$, $f'(0) = -1$ and $f''(0) = 2$. More than one answer may be correct.)

   ⊗ (i) ○ (ii) ○ (iii) ○ (iv) ○ None



(i) (ii) (iii) (iv)

   Solution. Since $f(0) = -0.5$ the graph passes through $(0, -0, 5)$, so the only two possibilities are (i) and (iii). Since $f'(0) = -1$, the function is decreasing at this point, so it must be figure (i).

2. Evaluate (find the sum of) the series $5 - \dfrac{10}{3} + \dfrac{20}{9} - \dfrac{40}{27} + \dfrac{80}{81} - \cdots$

   Solution. The series is geometric with $r = -2/3$ and $a = 5$. Since $|r| < 1$, the series converges, and its sum is $\frac{a}{1-r} = 3$.

3. Use the power series for e^x about $a = 0$ to find the power series for $z^2 e^{-z^3}$ about $a = 0$. Express your answer both in summation form, and by writing out the first four nonzero terms.

   Solution. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $e^{-z^3} = \sum_{n=0}^{\infty} \frac{(-z^3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{3n}}{n!}$, and thus $z^2 e^{-z^3} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{3n+2}}{n!}$. The first four terms are $z^2 - z^5 + \frac{1}{2}z^8 - \frac{1}{6}z^{11}$.

4. Determine whether the given series converges or diverges. **Justify your answers.**

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 10}$

   Solution. This series diverges by the integral test since

$$\int_1^{\infty} \frac{x^2}{x^3 + 10}\, dx = \lim_{b \to \infty} \frac{1}{3} \ln(x^3 + 10)\Big|_1^b = \infty$$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n^3 + 3n^2 + 5}$

Solution. Since $4n^3+3n^2+5 > 4n^3$, we have $\frac{n}{4n^3+3n^2+5} < \frac{n}{4n^3} = \frac{1}{4n^2}$. So since $\sum_{n=1}^{\infty} \frac{1}{4n^2}$ converges (it is a constant multiple of a p-series with $p = 2 > 1$), the series converges by the comparison test.

5. Let $f(x) = \sqrt{x}$. Use **the definition** to calculate its Taylor polynomial of degree 3 at $a = 1$.

Solution. $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, and $f'''(x) = \frac{3}{8}x^{-5/2}$. At $x = 1$ we have $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$ and $f'''(1) = \frac{3}{8}$. Therefore the 3rd degree Taylor polynomial at $a = 1$ is $P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$.

6. Both parts refer to the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 \cdot 3^n}$.

(a) Use the ratio test to find the radius of convergence.

Solution.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-2|n^2}{3(n+1)^2} = \frac{|x-2|}{3}$$

The ratio test implies convergence when $|x-2|/3 < 1$, so $-1 < x < 5$. The radius of convergence is 3.

(b) Investigate the endpoint behavior, and determine the interval of convergence.

Solution. At $x = -1$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. This is an alternating series and the terms $\frac{1}{n^2}$ decrease to zero, so the series converges.

At $x = 5$, the series is $\sum_{n=1}^{\infty} \frac{1}{n^2}$. This is a p-series with $p = 2 > 1$, so it converges.

The interval of convergence is therefore $-1 \leq x \leq 5$.

7. (a) Use the Comparison Test to determine whether or not

$$\sum_{n=0}^{\infty} \frac{n+3^n}{2^n}$$

converges.

Solution. Since $\frac{n+3^n}{2^n} \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ and the geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ diverges (because its ratio is $r = 3/2 > 1$), the series diverges.

(b) Use the Integral Test to determine whether or not

$$\sum_{k=0}^{\infty} \frac{k}{e^k}$$

converges.

Solution. Converges since

$$\int_0^{\infty} \frac{x}{e^x}\,dx = \int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty} \int_0^b xe^{-x}\,dx = \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_0^b = 1$$

(c) Use the Ratio Test to determine whether or not

$$\sum_{k=0}^{\infty} \frac{3^n}{n!}$$

converges.

Solution. Converges, since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3}{n+1}\right| = 0 < 1.$

8. Determine (with justification!) whether or not the following series converge:

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}, \qquad \sum_{n=0}^{\infty}(-1)^n\frac{n^2+4n+1}{3n^4+2n^2+10000}, \qquad \sum_{n=1}^{\infty}\frac{1}{n^{1.01}}.$$

Solution. The first series is a p-series with $p = 1/2 < 1$, so it diverges. The second series converges since it is alternating and the terms $\frac{n^2+4n+1}{3n^4+2n^2+10000}$ decrease to zero. The third series is a p-series with $p = 1.01 > 1$ so it converges.

9. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Find the 4th degree Taylor polynomial of f centered at $a = 0$. Find a factorial expression for the general term of the Taylor series.

Solution. $p_4(x) = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{15}{16\cdot 24}x^4$. The first few derivatives are $f'(0) = \frac{1}{2}$,

$$f^{(2)}(0) = -\frac{1}{2}\cdot\frac{1}{2} \qquad f^{(3)}(0) = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \qquad f^{(4)}(0) = -\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \qquad f^{(5)}(0) = \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}.$$

For $k \geq 2$, the general expression is

$$f^{(k)}(0) = (-1)^{k-1}\frac{(2k-3)}{2}\cdot\frac{(2k-5)}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = (-1)^{k-1}\frac{(2k-2)!}{2^k}\cdot\frac{1}{2^{k-1}(k-1)!}$$

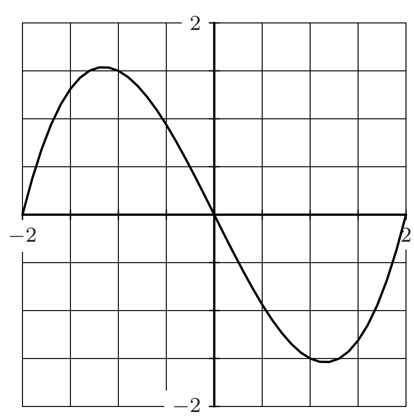
10. For each of the given power series, find the interval of convergence.

$$f(x) = \sum_{n=1}^{\infty}\frac{(2x)^n}{\sqrt{n}}, \qquad g(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-5)^n}{n\cdot 3^n}.$$

(In particular, give the radius of convergence, and investigate convergence at the endpoints.)

Solution. For $f(x)$, $-\frac{1}{2} \leq x < \frac{1}{2}$. For $g(x)$, $2 < x \leq 8$. (The first has radius $1/2$, the second has radius 3.)

11. The second degree Taylor polynomial of $f(x)$ at $a = 0$ is $p_2(x) = A + Bx + Cx^2$. What can you say about the signs of A, B, C if you know the graph of $f(x)$ is:



Solution. $A = f(1)$ is negative since the graph lies below the x-axis at $x = 1$, $B = f'(1)$ is also negative since the graph is decreasing at $x = 1$ and $C = \frac{1}{2}f''(1)$ is positive since the graph is concave up at $x = 1$.