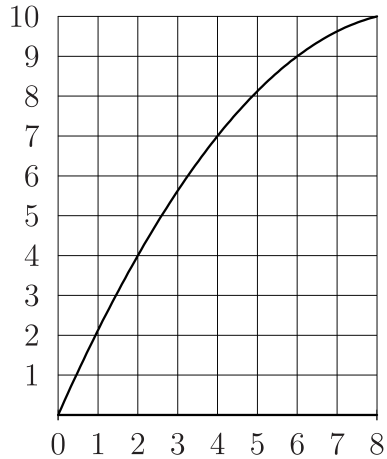
Math 132: Calculus for the Physical & Life Sciences 2
Spring 2006
Practice Questions for Midterm 2

1. The graph of the function $f(x)$ is shown below.



(a) Use the graph of f to compute the following approximations of $\int_0^8 f(x)\,dx$.

Solution: Since $n = 2$, $\Delta x = \frac{8-0}{2} = 4$.

$LEFT(2)$: $\underline{\ 28\ } = 0 \cdot 4 + 7 \cdot 4$

$RIGHT(2)$: $\underline{\ 68\ } = 7 \cdot 4 + 10 \cdot 4$

$MID(2)$: $\underline{\ 52\ } = 4 \cdot 4 + 9 \cdot 4$

$TRAP(2)$: $\underline{\ 48\ } = \frac{1}{2}(LEFT(2) + RIGHT(2))$

$SIMP(2)$: $\underline{\ 50.6\ } = \dfrac{2MID(2) + TRAP(2)}{3}$

(b) For each method, decide whether the approximation of $\int_0^8 f(x)\,dx$ is an *overestimate*, an *underestimate*, or that this *cannot be determined* from the given information. In the case of an overestimate or underestimate, briefly explain your reasoning.

Method	Type of Estimate	Reason
LEFT	underestimate	f is increasing
RIGHT	overestimate	f is increasing
MID	underestimate	f is concave down
TRAP	overestimate	f is concave down
SIMP	can't determine	

2. (a) Set up an integral that represents the arclength of the portion of the graph of $y = \sin(x)$ between $x = 0$ and $x = \pi$. **Do not compute the integral.**

Solution: Using the usual form of the arclength integral for the graph $y = f(x)$, we get $\frac{dy}{dx} = \cos(x)$ and

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^\pi \sqrt{1 + \cos^2 x}\,dx$$

(b) Approximate the integral in part (a) using a left hand sum with $n = 4$ subintervals.

Solution: We have $\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$. Then with the left hand sum approximation:

$$\begin{aligned}
\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx &\approx \sqrt{1+\cos^2(0)} \cdot \frac{\pi}{4} + \sqrt{1+\cos^2\left(\frac{\pi}{4}\right)} \cdot \frac{\pi}{4} \\
&\quad + \sqrt{1+\cos^2\left(\frac{\pi}{2}\right)} \cdot \frac{\pi}{4} + \sqrt{1+\cos^2\left(\frac{3\pi}{4}\right)} \cdot \frac{\pi}{4} \\
&= \left(\sqrt{2} + \sqrt{3/2} + 1 + \sqrt{3/2}\right) \frac{\pi}{4} \\
&= \left(\sqrt{2} + \sqrt{6} + 1\right) \frac{\pi}{4} \\
&\approx 3.82
\end{aligned}$$

3. Rewrite each of the following improper integrals as a limit, or limits. State whether the integral converges or diverges, and compute its value if it converges.

(a) $\displaystyle\int_{-1}^{2} \frac{1}{x^3}\, dx$

Solution: This integral is improper because the integrand is undefined at $x = 0$. Since this is in the interior of the interval of integration, we look at

$$\lim_{b \to 0^-} \int_{-1}^{b} x^{-3}\, dx + \lim_{a \to 0^+} \int_{a}^{2} x^{-3}\, dx = \lim_{b \to 0^-} \frac{-1}{2} x^{-2} \Big|_0^b + \lim_{a \to 0^+} \frac{-1}{2} x^{-2} \Big|_a^2$$

Both of these limits are undefined, so the improper integral *diverges*.

(b) $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$

Solution: This integral is improper because the integrand is not defined at the endpoint $x = 2$. We have

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx &= \lim_{b \to 2^-} \int_0^b \frac{1}{\sqrt{4-x^2}}\, dx \\
&= \lim_{b \to 2^-} \arcsin\left(\frac{x}{2}\right) \Big|_0^b \qquad \text{\#28 in table} \\
&= \lim_{b \to 2^-} \arcsin\left(\frac{b}{2}\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

So the integral converges to the value $\frac{\pi}{2}$.

(c) $\displaystyle\int_0^{\infty} e^{-4x}\, dx$

Solution: This integral is improper because of the infinite limit of integration.

$$\begin{aligned}\int_0^\infty e^{-4x}\,dx &= \lim_{b\to\infty}\int_0^b e^{-4x}\,dx \\ &= \lim_{b\to\infty} \frac{-1}{4}e^{-4x}\Big|_0^b \\ &= \lim_{b\to\infty}\frac{1}{4}(1-e^{-4b}) \\ &= \frac{1}{4}\end{aligned}$$

So the integral converges to the value $\frac{1}{4}$.

4. Use the comparison test to determine if each of the following improper integrals converges or diverges. You do not need to compute the value of the integral.

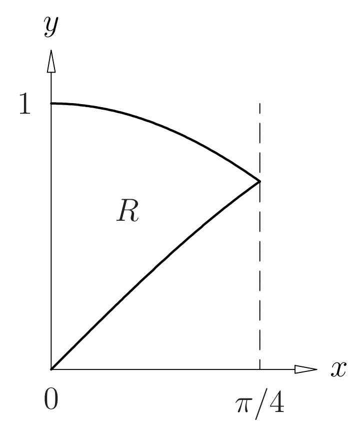
(a) $\displaystyle\int_1^\infty \frac{x^4}{x^5+1}\,dx$

Solution: For large x, the integrand behaves like $\frac{x^4}{x^5}=\frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, the integral diverges.

(b) $\displaystyle\int_1^\infty \frac{\cos(x)}{\sqrt{x^3+5}}\,dx$

Solution: For all x, $|\cos(x)|\le 1$ and for large x, $\sqrt{x^3+5}$ behaves like $\sqrt{x^3}=x^{3/2}$. Since $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, the integral converges.

5. Let R denote the region bounded by $y=\sin x$, $y=\cos x$, $x=0$ and $x=\pi/4$.



(a) Find the area of R.

Solution: Think of subdividing the area into vertical strips. The height of each is $\cos(x)-\sin(x)$ and the width is dx, so the area is given by

$$\int_0^{\pi/4}\cos(x)-\sin(x)\;dx = (\sin(x)+\cos(x))\Big|_0^{\pi/4} = \sqrt{2}-1\approx .414$$

(b) Find the volume of the solid obtained by revolving R about the x-axis.

Solution: Each cross-section is a washer with inner radius $\sin x$ and outer radius $\cos x$, so $A(x) = \pi(\cos^2 x - \sin^2 x)$ and the volume is

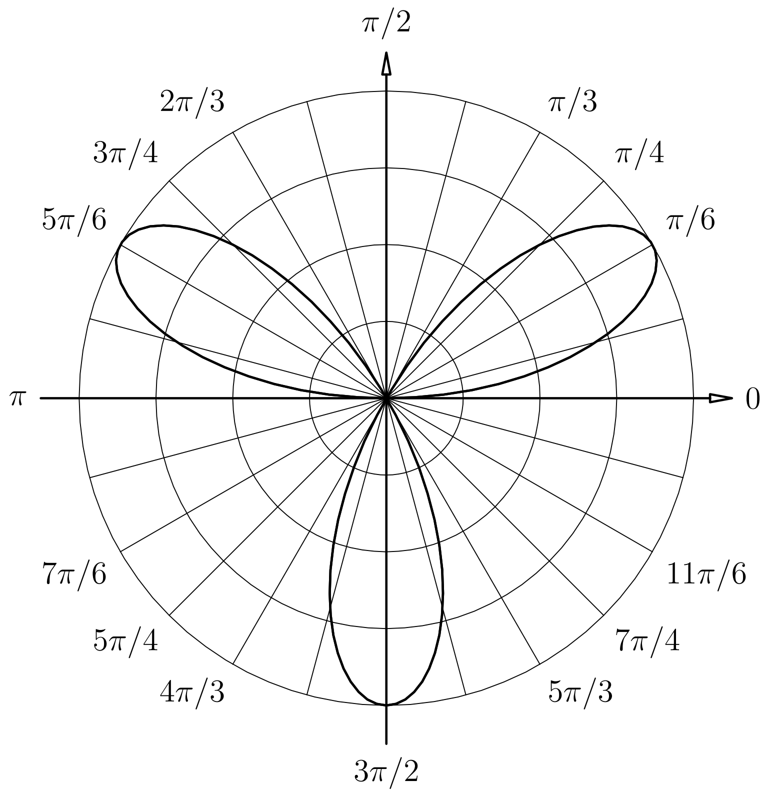
$$V = \int_0^{\pi/4} \pi(\cos^2 x - \sin^2 x)\, dx$$

To compute the integral, one could use the table of integrals, or the identity $\cos(2x) = \cos^2 x - \sin^2 x$. Either way, the value of the integral is $\frac{\pi}{2}$.

6. Consider the curve described in polar coordinates by the equation $r = \sin(3\theta)$.

(a) Sketch the curve.

Solution:



(b) Find the area of the region enclosed by one loop of the curve.

Solution: One loop of the curve begins at $\theta = 0$ and ends at $\theta = \pi/3$. Therefore its area is

$$A = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\, d\theta$$

Making the substitution $w = 3\theta$, this becomes

$$\frac{1}{6}\int_0^{\pi} \sin^2 w\, dw = \frac{\pi}{12}.$$

(c) Write the equation of the curve in Cartesian coordinates. Hint: Multiply the equation of the curve by r^3 and use the identity $\sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta$.

Solution: Multiplying by r^3 gives $r^4 = r^3\sin(3\theta)$, so using the identity, we get $r^4 = 3r^3\sin\theta\cos^2\theta - r^3\sin^3\theta$. Now using the fact that $r^2 = x^2 + y^2$, $x = r\cos\theta$ and $y = r\sin\theta$, we get $(x^2 + y^2)^2 = 3yx^2 - y^3$.

7. Suppose a metal rod of length 2 meters has a mass density $\delta(x) = 5 + 1.2x^2$ kg per meter.

   (a) Find the total mass of the rod.

   Solution: Since the rod is a straight thin (essentially 1-dimensional) object, the mass of a small piece is approximately density value times the length. Adding up these terms, we get:

$$m = \int_0^2 5 + 1.2x^2\ dx = (5x + .4x^3)\Big|_0^2 = 13.2\text{kg}$$

   (b) Find the center of mass of the rod.

   Solution: The center of mass is at

$$\begin{aligned}
\overline{x} &= \frac{1}{13.2}\int_0^2 x(5 + 1.2x^2)\ dx \\
&= \frac{1}{13.2}\int_0^2 5x + 1.2x^3\ dx \\
&= \frac{1}{13.2}\left(\frac{5x^2}{2} + .3x^4\Big|_0^2\right) \\
&= \frac{14.8}{13.2} \\
&\approx 1.12
\end{aligned}$$

8. The distribution of length in a certain population of inchworms is described by the pdf $p(x) = c(3x^2 - x^3)$ for $0 \le x \le 3$.

   (a) Find the value of the constant c.

   Solution: By the definition of a density function, we must have $\int_0^3 p(x)\,dx = 1$. Since

$$\int_0^3 p(x)\,dx = c\int_0^3 3x^2 - x^3\ dx = c(x^3 - \frac{1}{4}x^4)\Big|_0^3 = \frac{27c}{4}$$

   it follows that $c = \frac{4}{27}$. (b) Find the proportion of inchworms with length at most 2 cm.

   Solution:

$$\int_0^2 p(x)\,dx = \int_0^2 \frac{4}{27}(3x^2 - x^3)\,dx = \frac{4}{27}(x^3 - \frac{1}{4}x^4)\Big|_0^2 = \frac{16}{27}$$

9. Suppose

$$p(x) = \frac{4}{\pi}\cdot\frac{1}{1+x^2}$$

   is the probability density function for the quantity x, for $0 \le x \le 1$.

   (a) Find the mean of x.

Solution:

$$\mu = \int_0^1 xp(x)\,dx = \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2}\,dx = \frac{4}{\pi}\frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{2\ln 2}{\pi}$$

(b) Find the median of x.

Solution: The median T satisfies $\int_0^T p(x)\,dx = \frac{1}{2}$. Since

$$\int_0^T p(x)\,dx = \frac{4}{\pi}\int_0^T \frac{1}{1+x^2}\,dx = \frac{4\arctan(x)}{\pi}\Big|_0^T = \frac{4\arctan(T)}{\pi}$$

the median is $T = \tan(\pi/8)$.