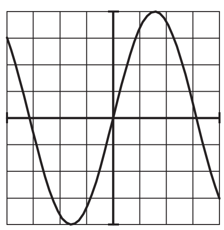
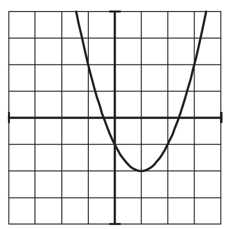
College of the Holy Cross, Fall Semester, 2005
Math 131, Practice Final Answers

1. Circle the number corresponding to the graph of each function. Each square in the figures is 1 unit by 1 unit, and the bold lines are the axes.

(a)	$\frac{1}{2}x - 2$	I	II	III	IV	$\boxed{\text{V}}$	VI	VII	VIII	IX
(b)	$\sin(4x)$	I	II	III	IV	V	VI	VII	$\boxed{\text{VIII}}$	IX
(c)	$x^3 - 2x - 1$	I	II	III	$\boxed{\text{IV}}$	V	VI	VII	VIII	IX
(d)	$\dfrac{x}{x-1}$	I	II	$\boxed{\text{III}}$	IV	V	VI	VII	VIII	IX
(e)	e^{-x}	I	II	III	IV	V	$\boxed{\text{VI}}$	VII	VIII	IX
(f)	$x^2 - 2x - 1$	I	$\boxed{\text{II}}$	III	IV	V	VI	VII	VIII	IX
(g)	$4\sin(x)$	$\boxed{\text{I}}$	II	III	IV	V	VI	VII	VIII	IX

(I)


(II)


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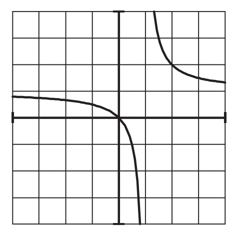

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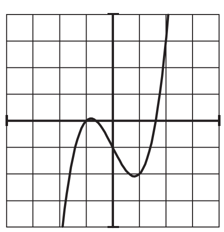

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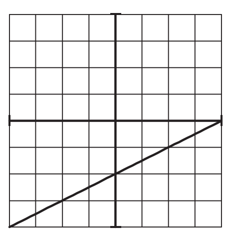

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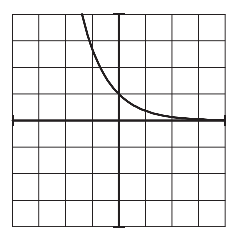

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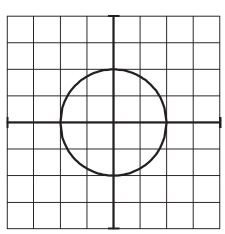
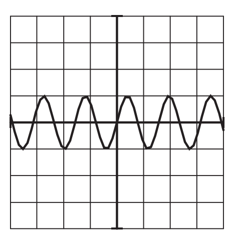
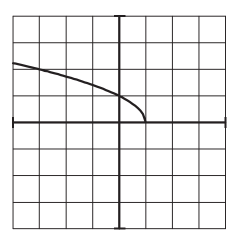


(VIII)


(IX)


2. Compute the following limits.

(a) $\lim_{x\to 0} \frac{x^2-9}{3-x}$

Answer. Since the numerator and denominator are continuous, and the denominator is nonzero at $x=0$,

$$\lim_{x\to 0} \frac{x^2-9}{3-x} = \frac{-9}{3} = -3.$$

(b) $\lim_{x\to 1} \frac{2^x-2}{\ln(x)}$,

Answer. The numerator and denominator both approach zero, so by L'Hopital's Rule,

$$\lim_{x\to 1} \frac{2^x-2}{\ln(x)} = \lim_{x\to 1} \frac{2^x \ln 2}{1/x} = 2\ln 2.$$

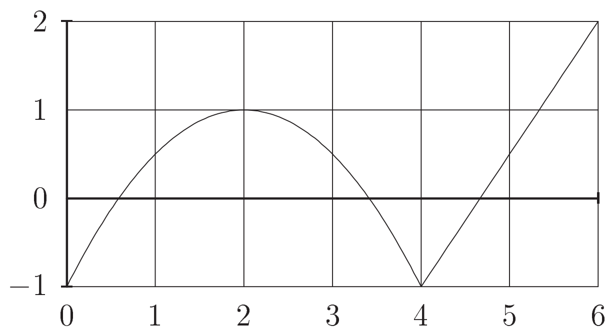
(c) $\lim_{x\to\infty} \frac{4x^2+3e^{-x}}{5x^2+7e^{-x}}$

Answer. The numerator and denominator both approach infinity, so by applying L'Hopital's Rule (twice), we get

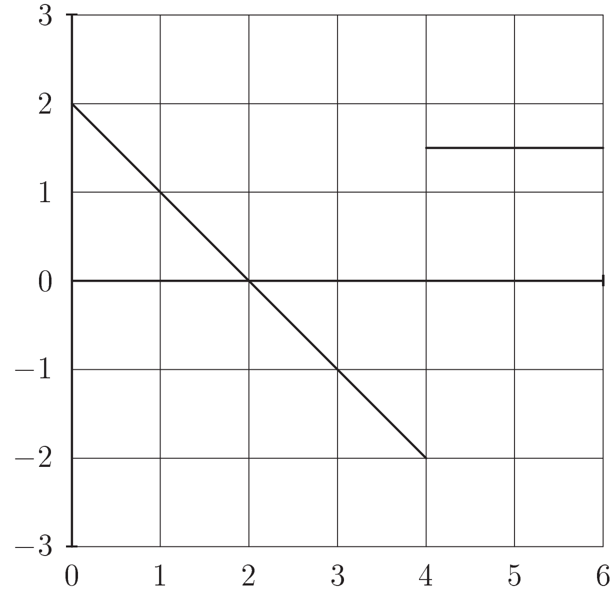
$$\lim_{x\to\infty} \frac{4x^2+3e^{-x}}{5x^2+7e^{-x}} = \lim_{x\to\infty} \frac{8x-3e^{-x}}{10x-7e^{-x}} = \lim_{x\to\infty} \frac{8+3e^{-x}}{10+7e^{-x}} = \frac{4}{5}$$

3. The graph of the function $f(x)$ is shown below. On the axes provided, sketch the graph of $f'(x)$. Be sure to label the points where f' is zero or undefined.

Graph of f:



Graph of f':



4. (a) Suppose $f(2) = 3$ and $f'(2) = -5$. Use the linear approximation of f at $x = 2$ to estimate the value of $f(1.97)$.

   Answer. The linear approximation of f at $x = 2$ is $f(x) \approx f(2) + f'(2)(x - 2) = 3 - 5(x - 2)$, so $f(1.97) \approx 3 - 5(1.97 - 2) = 3 - 5(-0.03) = 3.15$.

   (b) Suppose as in part (a) that $f(2) = 3$ and $f'(2) = -5$. Also suppose $g(3) = -1$ and $g'(3) = 7$. Let $h(x) = g(f(x))$. Find $h'(2)$.

   Answer. By the chain rule, $h'(x) = g'(f(x)) \cdot f'(x)$. At $x = 2$, $h'(2) = g'(f(2)) \cdot f'(2) = g'(3) \cdot (-5) = 7(-5) = -35$.

5. Compute the derivative of each function.

   (a) $f(x) = \dfrac{x^3}{2x^2 + 1}$

   Answer. By the quotient rule,

$$f'(x) = \frac{(2x^2 + 1)(3x^2) - x^3(4x)}{(2x^2 + 1)^2} = \frac{x^2(2x^2 + 3)}{(2x^2 + 1)^2}.$$

   (b) $g(x) = x^2 \cos(3x)$

   Answer. By the product and chain rules,

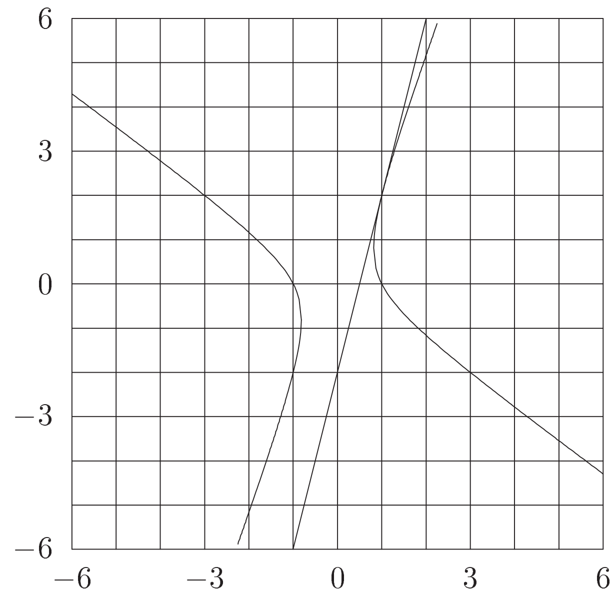
$$g'(x) = -3x^2 \sin(3x) + 2x \cos(3x) = x(-3x \sin(3x) + 2 \cos(3x)).$$

   (c) $h(x) = \ln(1 + e^{\sqrt{x}})$

   Answer. By the chain rule,

$$h'(x) = \frac{1}{1 + e^{\sqrt{x}}} \cdot e^{\sqrt{x}} \cdot \frac{1}{2} x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}(1 + e^{\sqrt{x}})}.$$

6. The hyperbola $2x^2 + 2xy - y^2 = 2$ is shown below.



(a) Compute $\dfrac{dy}{dx}$ in terms of x and y.

Answer. Differentiating with respect to x gives

$$4x + 2x\frac{dy}{dx} + 2y - 2y\frac{dy}{dx} = 0 \qquad \Longrightarrow \qquad \frac{dy}{dx} = \frac{2x+y}{y-x}$$

(b) Find the equation of the tangent line to the curve at the point $(1, 2)$ and sketch the tangent line on the figure above.

Answer. At $(1, 2)$, $\frac{dy}{dx} = 4$, so the equation of the tangent line is $y - 2 = 4(x - 1)$, or $y = 4x - 2$.

7. Suppose the position of an object is described by the parametric curve $x = t^3 - t^2$, $y = 3t^3 - 4t$, where t is measured in seconds.

(a) Find the location of the object at time 2 seconds. Find the equation of the tangent line to the curve at this point.

Answer. The object is located at $x = 4$, $y = 16$. The tangent line has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2 - 4}{3t^2 - 2t} = \frac{32}{8} = 4$$

when $t = 2$. Therefore the equation of the tangent line is $y - 16 = 4(x - 4)$, or $y = 4x$.

(b) At what time is the speed of the object zero?

Answer. The speed is zero when both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero. Since $\frac{dx}{dt} = 3t^2 - 2t = 0$ when $t = 0$ and $t = -2/3$, and $\frac{dy}{dt} = 9t^2 - 4 = 0$ when $t = \pm\frac{2}{3}$, the speed is zero only when $t = \frac{2}{3}$.

8. Let $f(x) = x^3 e^x$.

(a) Find and classify (local min/max, or neither) the critical points of f.

Answer. By the product rule, $f'(x) = 3x^2e^x + x^3e^x = e^x x^2(3+x)$, so the critical points of f are $x = 0$ and $x = -3$. Since $f'(x) < 0$ for $x < -3$, $f'(x) > 0$ for $-3 < x < 0$ and $f'(x) > 0$ for $x > 0$, the first derivative test implies that f has a local minimum at $x = -3$, but neither a local max nor a local min at $x = 0$.

(b) Find the inflection points of f.

Answer. By the product rule again, $f''(x) = e^x(6x + 3x^2) + e^x(3x^2 + x^3) = xe^x(6 + 6x + x^2)$. The roots of f'' are therefore $x = 0$ and $x = -3 \pm \sqrt{3}$. These are all inflection points since f'' changes sign at each of them.

9. You have \$3 with which to construct a box with a square base. The material for the top and bottom costs 2 cents per square inch, while the material for the sides costs 1 cent per square inch. What dimensions maximize the volume of the box? *Hint:* First write the cost of the box and the volume of the box in terms of the dimensions of the box.

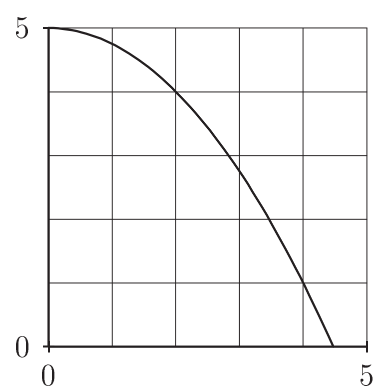
Answer. Let x denote the side length of the square, and y the height. Then the volume is $V = x^2y$ and the cost of the box is $C = 2(2x^2) + 1(4xy)$. Since we have \$3 to use, this means $4x^2 + 4xy = 300$, and solving for y gives $y = (300 - 4x^2)/(4x)$. Therefore the volume is $V(x) = \frac{1}{4}x(300 - 4x^2) = \frac{1}{4}(300x - 4x^3)$. The domain is $0 \le x \le \sqrt{300/4}$. Solving $V'(x) = \frac{1}{4}(300 - 12x^2) = 0$ gives the critical point $x = 5$ ($x = -5$ is not in the domain). Since $V(0) = V(\sqrt{300/4}) = 0$, the maximum volume is $V(5) = 250$, and the dimensions that produce this volume are $x = 5$ and $y = 10$.

10. Suppose you are making a snow-person, and you begin by rolling a snowball in the snow. Assume the snowball always remains in the shape of a sphere. If the radius of the snowball increases at a constant rate of 2 inches per second, how fast is the volume of the snowball increasing when the radius is 6 inches? Give the units of your answer. The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Answer. We are given that $\frac{dr}{dt} = 2$ and want to find $\frac{dV}{dt}$ when $r = 6$. Differentiating the volume formula gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{sec}$$

11. The graph of $f(x) = 5 - \frac{1}{4}x^2$ is shown below.



(a) Compute the left and right hand sums for f over the interval $1 \le x \le 4$ using $n = 3$ subintervals. On the figure above sketch the rectangles which represent the terms in the left and right hand sums.

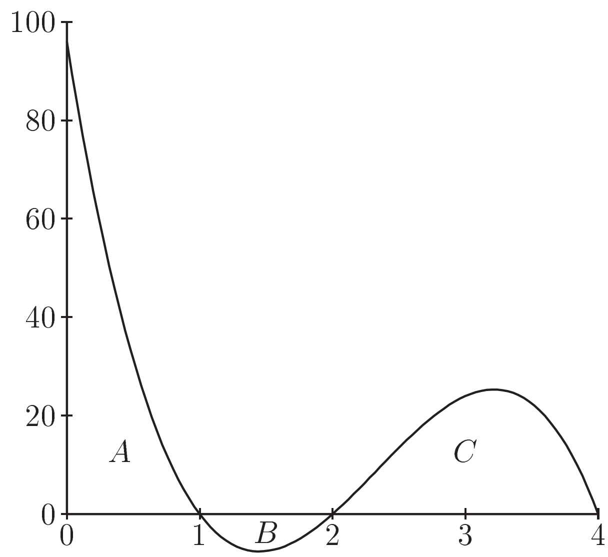
Answer. $\Delta x = 1$ so

$$\begin{aligned} \text{LHS} &= f(1) + f(2) + f(3) = (5 - \tfrac{1}{4}) + (5 - 1) + (5 - \tfrac{9}{4}) = 11.5 \\ \text{RHS} &= f(2) + f(3) + f(4) = (5 - 1) + (5 - \tfrac{9}{4}) + (5 - 4) = 7.75 \end{aligned}$$

(b) Which sum is a better estimate of the exact area under the graph of f over $1 \le x \le 4$? Explain.

Answer. Because the graph of f is decreasing and concave down, the left hand sum gives a better estimate.

12. The graph of $f(x)$ is shown below. The areas of the regions shown are $A = 37$, $B = 5$ and $C = 32$.



(a) Evaluate $\int_0^4 f(x)\,dx$

Answer. $\int_0^4 f(x)\,dx = A - B + C = 64.$

(b) Find the average value of f over the interval $1 \le x \le 4$.

Answer. The average value is

$$\frac{1}{4-1}\int_1^4 f(x)\,dx = \frac{1}{3}(-B + C) = \frac{1}{3}(-5 + 32) = 9.$$

13. Suppose the rate of growth of a population of bats living under a bridge, in thousands of bats per year, is

$$f(t) = 2 + \frac{t}{3}$$

where t is measured in years since 1990. Suppose there were 40 thousand bats in 1990.

(a) Write out an integral that represents the change in the bat population from 1990 to 2005.

Answer. Let $F(t)$ be the population (in thousands) in year t. Then $F'(t) = f(t)$, so by the Fundamental Theorem of Calculus,

$$F(15) - F(0) = \int_0^{15} 2 + \frac{1}{3}t\, dt$$

is the change in the bat population from 1990 to 2005.

(b) Compute the integral exactly (use the graph of f) and determine what the population will be in 2005.

Answer. Since f is a linear function, the region under its graph is the union of a rectangle and a triangle. The total area under the graph is $15(2)+\frac{1}{2}(15)(5) = 67.5$, so

$$\int_0^{15} 2 + \frac{1}{3}t\, dt = 67.5.$$

Solving for $F(15)$ gives $F(15) = F(0) + 67.5 = 40 + 67.5 = 107.5$, which means there are $107,500$ bats in 2005.