Calculus for the Phys./Life Sciences 1

Sept. 10, 2002

Worksheet on Compound Interest and The Special Number e

Worksheet Write-ups Due Monday Sept. 16 at the start of class.

This worksheet is intended to introduce the very important number e through a particular real-world application, namely compound interest. When you invest in a mutual fund or deposit your money into a bank account or CD, you expect to make a certain rate of return r on your money. Your money can be **compounded** at different times during the year. Compounded yearly means that after one year, you make r percent of your initial investment and add it to your original. (This was covered in class with exponential functions in Section 1.2.) Compounded **quarterly** means that the bank adjusts the rate to $r/4$, but compounds your money 4 times a year (usually it is every 3 months). Compounded **monthly** means a rate of $r/12$ compounded 12 times a year. In other words, the more your money is compounded, the lower the rate (seems bad), but the more often its compounded (seems good). So which is better, compounded yearly or quarterly?

Answer the following questions neatly and carefully. Be sure to round to the nearest cent in your dollar answers, but do not round off until the answer is obtained. Rounding off too soon can lead to numerical error.

1. Suppose you start with an initial investment of \$10,000 at a rate r of 8%. Find the amount P in your account after 1 year if it is compounded yearly versus if it is compounded quarterly. Find the amount in your account using each method after 10 years. What is the formula $P(t)$, where t is time in years, for the amount of money in your account if its compounded yearly and if its compounded quarterly. Which method yields the greater value?

2. What is the general formula for $P(t)$ if P_0 is the initial quantity invested at an interest rate r after t years, if it is compounded yearly? compounded quarterly?

3. Repeat the above two questions for compound interest monthly and compound interest daily (365 times a year)? Which is the best after 10 years?

4. Compute the bases to 7 decimal places of the exponential functions in the above examples (yearly, quarterly, monthly and daily with interest rate $r = 8\%$). In other words, write each of the above functions in the standard form $P(t) = P_0 a^t$. The base is the value of a. Compare the bases to each other. Which is the largest? Why does this make sense based on your findings in questions 1 and 3?

5. What is the general formula for the amount of an investment $P(t)$ if the initial amount is P_0, the interest rate is r and the account is compounded m times a year? Assume t is in years.

6. What would the base of the exponential function (to 7 decimal places) be if the account was compounded 1000 times a year? 10,000 times a year? (interest rate of 8%) Compare your answers to the value of the number $e^{0.08}$. What do you notice?

7. The term used for compounding an investment at every possible instant over the course of a year is **compounded continuously**. Picture a very fast mathematician calculating your account over and over again without ever taking a break. The formula for compound interest is

$$P(t) = P_0 e^{rt}$$

where P_0 is the initial amount, r is the interest rate and t is in years. How much money is in the above account ($10,000 invested at a rate of 8%) if it is compounded continuously for 1 year? for 10 years? Compare to your previous answers.

8. Based on your work thus far, **explain** why e can be defined as the following:

$$\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = e$$

Here, the "lim" notation means that we are plugging in bigger and bigger m values into the expression until the result seems to approach a fixed value. In this case, that value is the special number e. *Hint:* Try setting $r = 1$ and comparing the formula for interest compounded m times a year with the formula for compounding interest continuously. What is the base in each case?