MATH 126 Calculus for the Social Sciences II

Exam #3 April 24, 2008 Prof. G. Roberts

SOLUTIONS

1. Find the average value of the function

$$f(x) = \frac{x}{\sqrt{2x^2+1}}$$

over the interval $0 \leq x \leq 2$. (10 pts.)

Answer: By definition, the average value is

$$\frac{1}{2-0}\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx\ .$$

This integral can be done with a u-sub letting $u = 2x^2+1$ and thus $du = 4x\,dx$ or $(1/4)\,du = x\,dx$. The integral then becomes

$$\begin{aligned}
\frac{1}{2}\int_0^2 \frac{x}{\sqrt{2x^2+1}}\,dx &= \frac{1}{2}\cdot\frac{1}{4}\int \frac{1}{\sqrt{u}}\,du \\
&= \frac{1}{8}\int u^{-1/2}\,du \\
&= \frac{1}{8}\cdot 2u^{1/2} \\
&= \frac{1}{4}\sqrt{2x^2+1}\,\Big|_0^2 \\
&= \frac{1}{4}(3-1) \;=\; \frac{1}{2}\ .
\end{aligned}$$

2. A demand curve for a product is given by $p(x) = \dfrac{600}{x+4}$. Find the consumer surplus (to the nearest cent) when the price level is $\bar{p}$ =\$15. (12 pts.)

Answer: First, find the quantity $\bar{x}$ when $\bar{p}$ =\$15 by solving

$$15 = \frac{600}{x+4}$$

for x. Cross-multiplying yields $15x + 60 = 600$ or $15x = 540$ and thus $x = 36$. By definition of consumer surplus, we must find

$$\begin{aligned}
\int_0^{36} \frac{600}{x+4} - 15\,dx &= 600\ln|x+4| - 15x\big|_0^{36} \\
&= 600\ln(40) - 540 - (600\ln(4) - 0) \\
&= \$841.55\ .
\end{aligned}$$

Note: It is important to integrate term by term here and **NOT** to factor out the 600 from the first term. The 600 is only multiplying the first term and not the second. It is ok to pull out a 15, for example, because 15 divides into each term evenly.

3. Let $f(t) = \frac{5}{8}t^3(2-t)$ if $0 \le t \le 2$ and $f(t) = 0$ if $t < 0$ or $t > 2$ be the probability density function for the amount of time spent waiting in line for a game-day Red Sox ticket, where t is measured in hours.

(a) Verify that $f(t)$ is a probability density function. (10 pts.)

Answer: First, we must check that $f(t) \ge 0$ for all t. It is clear that $f(t) \ge 0$ for $t < 0$ or $t > 2$ because $f(t) = 0$ for these values. On the interval $0 \le t \le 2$, the term t^3 is non-negative and so is the term $2 - t$. Thus, their product is also non-negative and we have that $f(t) \ge 0$ for $0 \le t \le 2$.

Secondly, we must verify that

$$\int_{-\infty}^{\infty} f(t)\, dt = 1.$$

Since $f(t) = 0$ for $t < 0$ or $t > 2$, this is equivalent to showing that

$$\int_0^2 \frac{5}{8}t^3(2-t)\, dt = 1.$$

The easiest way to do the integral is to **distribute** the t^3 term and then find an anti-derivative using the power rule. It is also possible to do the integral using integration by parts setting $u = 2 - t$ and $dv = t^3\, dt$.

$$\begin{aligned}
\int_0^2 \frac{5}{8}t^3(2-t)\, dt &= \frac{5}{8}\int_0^2 2t^3 - t^4\, dt \\
&= \frac{5}{8}\left(\frac{t^4}{2} - \frac{t^5}{5}\bigg|_0^2\right) \\
&= \frac{5}{8}\left(8 - \frac{32}{5} - 0\right) = \frac{5}{8}\cdot\frac{8}{5} = 1.
\end{aligned}$$

(b) What is the probability of waiting in line for **over** an hour to get tickets? (7 pts.)

Answer:

$$\begin{aligned}
P(t > 1) &= \int_1^{\infty} f(t)\, dt \\
&= \int_1^2 \frac{5}{8}t^3(2-t)\, dt \\
&= \frac{5}{8}\int_1^2 2t^3 - t^4\, dt \\
&= \frac{5}{8}\left(\frac{t^4}{2} - \frac{t^5}{5}\bigg|_1^2\right) \\
&= \frac{5}{8}\left(8 - \frac{32}{5} - \left(\frac{1}{2} - \frac{1}{5}\right)\right) \\
&= \frac{5}{8}\left(\frac{8}{5} - \frac{3}{10}\right) \\
&= \frac{5}{8}\cdot\frac{13}{10} = \frac{13}{16} = 0.8125 \text{ or } 81.25\%\,.
\end{aligned}$$

4. Find the possible values of k for which $y = e^{kx}$ is a solution to $y'' - 3y' - 10y = 0$. (10 pts.)

Answer: Plug it in. We have $y = e^{kx}$, $y' = ke^{kx}$ and $y'' = k^2 e^{kx}$ using the chain rule for each derivative. Substituting these into the ODE yields

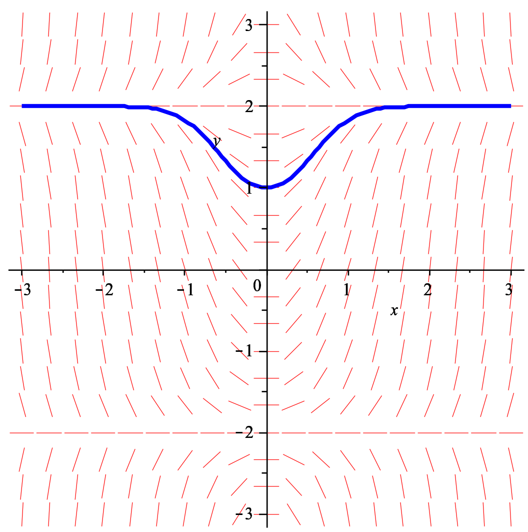
$$k^2 e^{kx} - 3ke^{kx} - 10e^{kx} \; = \; 0$$

or

$$e^{kx}\left(k^2 - 3k - 10\right) \; = \; 0.$$

Since the first term is an exponential, it is always positive. Thus, the only way to obtain 0 is for $k^2 - 3k - 10 = 0$. Factoring this quadratic as $(k-5)(k+2)$ leads to the solution $k = 5, -2$. Note that the only equilibrium solution to the ODE is simply $y = 0$. There is no k value that will give $y = 0$ unless you allow for $k = -\infty$.

5. The direction field for the ordinary differential equation (ODE) $y' = x(4 - y^2)$ is given below.



(a) What are the equilibrium solutions? (5 pts.)

Answer: From the direction field we see that the slope marks are horizontal when $y = 2$ and $y = -2$. From the ODE, we see that $y' = 0$ whenever $x = 0$ or $y = \pm 2$. However, $x = 0$ is a vertical line through the center of the direction field and **not** a function. Solutions to this ODE must be of the form $y = y(x)$. Thus, the two equilibrium solutions are $y(x) = 2$ and $y(x) = -2$.

(b) Use Euler's method with a step-size of $\Delta x = 0.25$ to approximate $y(1)$ where $y(x)$ is the solution to the given ODE with initial condition $y(0) = 1$. Give your approximation to 6 decimal places. (10 pts.)

Answer: We repeatedly apply the formulas

$$\begin{aligned} x_{n+1} &= x_n + 0.25 \\ y_{n+1} &= y_n + f(x_n, y_n) \cdot 0.25 \, . \end{aligned}$$

The slope m is found at each stage by plugging the current x and y-values into the right-hand side of the ODE: $x(4 - y^2)$.

n	x	y	m
0	0	1	0
1	0.25	1	0.75
2	0.5	1.1875	1.294921875
3	0.75	1.511230469	1.287136853
4	1	1.833014682	

Therefore, our Euler's method approximation gives $y(1) \approx 1.833015$.

(c) Using the direction field above, find $\lim_{x\to\infty} y(x)$ where $y(x)$ is the solution to the ODE with initial condition $y(0) = 1$. (5 pts.)

Answer: The solution curve $y(x)$ passing through the initial condition $y(0) = 1$ is shown on the direction field at the start of the problem. Since $y(x) = 2$ is also a solution to the ODE, our solution cannot cross $y = 2$ but does approach it as $x \to \infty$. Therefore, we have that

$$\lim_{x\to\infty} y(x) \;=\; 2\,.$$

6. Find the solution to the differential equation below satisfying the given initial condition. (13 pts.)

$$\frac{dy}{dx} \;=\; y^3\cos(2x)\,, \quad y(0) = \frac{1}{3}$$

Answer: Separate, integrate, solve for y and then find the particular value of the integration constant c so that $y(0) = 1/3$. We have

$$\begin{aligned}
\int \frac{dy}{y^3} &= \int \cos(2x)\; dx \\
\int y^{-3}\; dy &= \int \cos(2x)\; dx \\
\frac{y^{-2}}{-2} &= \frac{1}{2}\sin(2x) + c \quad (u\text{-sub with } u = 2x) \\
y^{-2} &= -\sin(2x) + c \\
\frac{1}{y^2} &= -\sin(2x) + c \\
y^2 &= \frac{1}{-\sin(2x) + c} \\
y &= \pm\sqrt{\frac{1}{-\sin(2x) + c}}
\end{aligned}$$

Since $y(0) = \frac{1}{3}$, we must choose the $+$ instead of the $-$ in front of the square root. Substituting $x = 0$ and $y = 1/3$ into our expression for y yields

$$\frac{1}{3} = \sqrt{\frac{1}{-\sin(0) + c}} \quad\Longrightarrow\quad \frac{1}{9} = \frac{1}{c} \quad\Longrightarrow\quad c = 9\,.$$

Therefore, the solution is

$$y = \sqrt{\frac{1}{9 - \sin(2x)}} \quad \text{or} \quad y = \frac{1}{\sqrt{9 - \sin(2x)}}.$$

7. Suppose that Aunt Julie is cooking her Thanksgiving turkey (tofurkey for you vegetarians) for friends and family. The guests are planning to arrive at 5:00 pm. She pre-heats the oven to 400°F. Suppose the initial temperature of the turkey is 65°F. She places the turkey in the oven at 10:00 am. At 1:00 pm the turkey has cooked to a temperature of 110°F. Using Newton's law of cooling (or warming), at what time (to the nearest minute) will the temperature of the turkey be 150°F (medium rare and ready to serve)? Assume that the oven has a constant temperature of 400°F throughout the cooking. Does she make it in time for the guests or will she be serving hors d'ouvres for a while? (18 pts.)

Answer: The ambient temperature is 400°F. We will let $t = 0$ correspond to 10:00 am and take t to be in hours. Letting $y(t)$ be the temperature of the turkey at time t, we have the following model using Newton's Law of Cooling:

$$\frac{dy}{dt} = k(y - 400), \quad y(0) = 65, \quad y(3) = 110$$

First we solve the ODE. Separate and integrate:

$$\frac{dy}{y - 400} = k\,dt \quad \Longrightarrow \quad \ln|y - 400| = kt + c \quad \Longrightarrow \quad |y - 400| = ce^{kt} \quad \Longrightarrow \quad y = 400 + ce^{kt}.$$

The first initial condition $y(0) = 65$ gives

$$65 = 400 + ce^0 \quad \Longrightarrow \quad c = -335$$

while the condition $y(3) = 110$ implies that

$$110 = 400 - 335e^{3k} \quad \Longrightarrow \quad -290 = -335e^{3k} \quad \Longrightarrow \quad \frac{290}{335} = e^{3k}$$

which gives

$$k = \frac{1}{3}\ln\left(\frac{58}{67}\right) \approx -0.048083203.$$

Thus, our function for the temperature of the turkey is

$$y = 400 - 335e^{-0.048083203t}.$$

The turkey is "cooked" when $y = 150$°F. Solving for t in the equation $y(t) = 150$ yields

$$150 = 400 - 335e^{kt} \quad \Longrightarrow \quad -250 = -335e^{kt} \quad \Longrightarrow \quad \frac{250}{335} = e^{kt}$$

which gives

$$t = \frac{1}{k}\ln\left(\frac{50}{67}\right) \approx 6.086732907 \approx 6 \text{ hours, } 5 \text{ minutes}.$$

Therefore, the turkey will be ready at 4:05 pm, in time for the guests!