MATH 126 Calculus for the Social Sciences II

Exam #2 March 27, 2008 Prof. G. Roberts

SOLUTIONS

1. Consider the definite integral $\int_0^2 \cos(x^2)\,dx$.

(a) Approximate the integral using the Trapezoid Rule with $n = 4$ subdivisions. Round your answer to 4 decimal places. (8 pts.)

Note: You must set your calculator to RADIANS to do these problems. (When in doubt, use radians, always.) Radians are actual physical units that have meaning. Degrees are not. Recall that the derivative of $\sin x$ is NOT $\cos x$ if you are in degrees! Radians is always assumed when doing trig. Always!

Answer: Let $f(x) = \cos(x^2)$. We have $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. Then

$$\begin{aligned}
\text{Left-hand Sum} &= \frac{1}{2}\left(f(0) + f(0.5) + f(1) + f(1.5)\right) \\
&= \frac{1}{2}\left(\cos(0) + \cos(0.25) + \cos(1) + \cos(2.25)\right) \\
&= 0.940520552
\end{aligned}$$

Note: This entire calculation can be done on the calculator, avoiding round-off error, even using the little TI-30's. Set the calculator to RADIANS. For the TI-30XA, you type 0, then x^2, then cos, then +, then 0.5, then x^2, then cos, then +, etc. When you are done, hit = and divide by 2.

$$\begin{aligned}
\text{Right-hand Sum} &= \frac{1}{2}\left(f(0.5) + f(1) + f(1.5) + f(2)\right) \\
&= \frac{1}{2}\left(\cos(0.25) + \cos(1) + \cos(2.25) + \cos(4)\right) \\
&= 0.113698742
\end{aligned}$$

Finally, we have that TRAP $= \frac{1}{2}(\text{LHS} + \text{RHS}) = 0.5271$

(b) Approximate the integral using the Midpoint Rule with $n = 4$ subdivisions. Round your answer to 4 decimal places. (5 pts.)

Answer:

$$\begin{aligned}
\text{Midpoint Sum} &= \frac{1}{2}\left(f(0.25) + f(0.75) + f(1.25) + f(1.75)\right) \\
&= \frac{1}{2}\left(\cos(0.0625) + \cos(0.5625) + \cos(1.5625) + \cos(3.0625)\right) \\
&= 0.4277
\end{aligned}$$

(c) Approximate the integral using Simpson's Rule with $n = 4$ subdivisions. Round your answer to 4 decimal places. (5 pts.)

Answer:

$$\begin{aligned}
\text{Simpson's Rule} &= \frac{1}{3}\text{TRAP} + \frac{2}{3}\text{MID} \\
&= 0.4608
\end{aligned}$$

2. Evaluate the definite integral $\int_0^{1/3} xe^{3x}\,dx$.   (10 pts.)

Answer: Use integration by parts.

Let $u = x$ and $dv = e^{3x}\,dx$. Then $du = 1dx$ and $v = \frac{1}{3}e^{3x}$ (from memorization or doing a u-sub). Then we have

$$\begin{aligned}
\int_0^{1/3} xe^{3x}\,dx &= \frac{1}{3}xe^{3x}\Big|_0^{1/3} - \int_0^{1/3}\frac{1}{3}e^{3x}\,dx \\
&= \frac{1}{3}xe^{3x}\Big|_0^{1/3} - \frac{1}{9}e^{3x}\Big|_0^{1/3} \\
&= \frac{1}{3}e^{3x}\left(x-\frac{1}{3}\right)\Big|_0^{1/3} \\
&= 0-\left(-\frac{1}{9}\right) = \frac{1}{9}
\end{aligned}$$

3. Use the substitution $x = 2\tan\theta$ to evaluate the integral

$$\int \frac{x^2}{x^2+4}\,dx\ .$$

Hint: You will need to use the identity $\tan^2\theta + 1 = \sec^2\theta$ **twice**. (10 pts.)

Answer: Start with $x = 2\tan\theta$ and differentiate to obtain $dx = 2\sec^2\theta\,d\theta$. Then substitute into the integral **everywhere** you have an x term.

$$\begin{aligned}
\int \frac{x^2}{x^2+4}\,dx &= \int \frac{4\tan^2\theta}{4\tan^2\theta+4}\cdot 2\sec^2\theta\,d\theta \\
&= \int \frac{4\tan^2\theta}{4(\tan^2\theta+1)}\cdot 2\sec^2\theta\,d\theta \\
&= \int \frac{4\tan^2\theta}{4\sec^2\theta}\cdot 2\sec^2\theta\,d\theta \\
&= 2\int \tan^2\theta\,d\theta \\
&= 2\int \sec^2\theta - 1\,d\theta \qquad \text{Using the hint a second time!} \\
&= 2(\tan\theta-\theta)+c \\
&= 2\left(\frac{x}{2}-\tan^{-1}\left(\frac{x}{2}\right)\right)+c \\
&= x-2\tan^{-1}(\frac{x}{2})+c
\end{aligned}$$

4. Consider the two integrals below. One of these can be integrated using a u-substitution while the other requires partial fractions. Determine which is which and evaluate **both** integrals. (18 pts.)

(a) $\displaystyle\int \frac{2x+9}{x^2-x-6}\,dx$ **(b)** $\displaystyle\int \frac{2x-1}{x^2-x-6}\,dx$

Answer: The first can be done with partial fractions while the second is a straight-forward u-sub. For integral **(a)**, factor the denominator as $x^2-x-6=(x-3)(x+2)$ and then break the fraction into two pieces:

$$\frac{2x+9}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2}$$

Multiply both sides by the LCD $(x-3)(x+2)$ to obtain the equation $2x+9=A(x+2)+B(x-3)$. Substituting in the root $x=-2$ into this equation gives $5=-5B$ and thus $B=-1$. Substituting the root $x=3$ into this equation gives $15=5A$ and thus $A=3$. Returning to our integral, we have

$$\int \frac{2x+9}{x^2-x-6}\,dx = \int \frac{3}{x-3}+\frac{-1}{x+2}\,dx = 3\ln|x-3|-\ln|x+2|+c$$

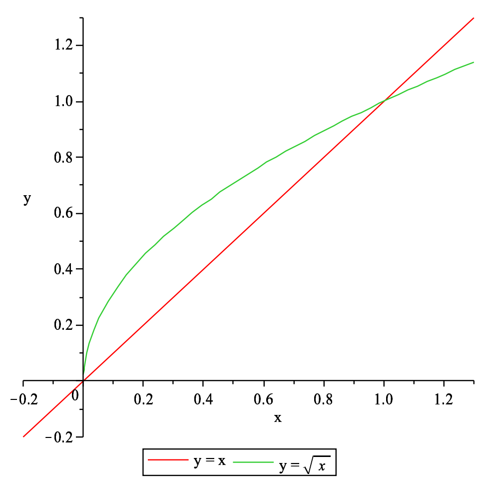
For integral **(b)**, let $u=x^2-x-6$. Then $du=2x-1\,dx$ and the integral transforms nicely into

$$\int \frac{1}{u}\,du = \ln|u|+c = \ln|x^2-x-6|+c$$

5. Let R be the region in the first quadrant bounded by the curves $y=x$ and $y=\sqrt{x}$.

(a) Sketch the region R and find its area. (8 pts.)

Answer: The graph of $y=x$ is just the diagonal through the first quadrant while the graph of $y=\sqrt{x}$ is the upper half of the right-facing parabola $x=y^2$. Note that $y=\sqrt{x}$ will be **above** the diagonal for $0<x<1$ and that the curves intersect at the points $(0,0)$ and $(1,1)$.



The area of the region is

$$A = \int_0^1 \sqrt{x} - x \, dx = \int_0^1 x^{1/2} - x \, dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^2}{2}\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

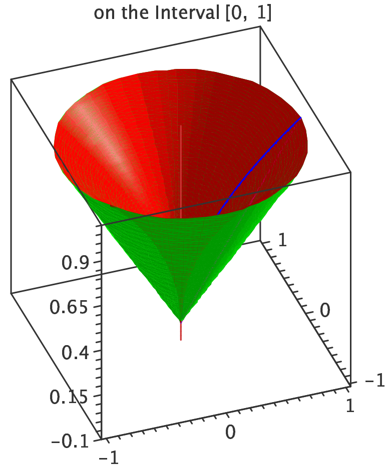
(b) Find the volume of the solid obtained by rotating R about the y-axis. (8 pts.)

Answer: Use the washer method and integrate with respect to y since we are rotating about the y-axis. The outer radius is y and the inner radius is y^2. The volume is then

$$V = \pi \int_0^1 (y)^2 - (y^2)^2 \, dy = \pi \int_0^1 y^2 - y^4 \, dy$$

$$= \pi \left(\frac{y^3}{3} - \frac{y^5}{5} \right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}$$

The Volume of Revolution Around the Vertical Axis Between
f(x) = x^(1/2)
and
g(x) = x
on the Interval [0, 1]

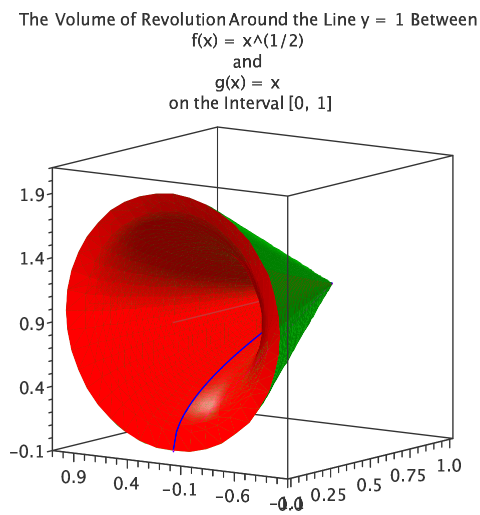


(c) Find the volume of the solid obtained by rotating R about the line $y = 1$. (8 pts.)

Answer: Use the washer method and integrate with respect to x since we are rotating about a line parallel to the x-axis. The outer radius is $1 - x$ and the inner radius is $1 - \sqrt{x}$. The volume is then

$$V = \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2 \, dx = \pi \int_0^1 1 - 2x + x^2 - (1 - 2\sqrt{x} + x) \, dx$$

$$= \pi \int_0^1 x^2 - 3x + 2\sqrt{x} \, dx = \pi \left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4x^{3/2}}{3} \right)\Big|_0^1 = \pi \left(\frac{5}{3} - \frac{3}{2} \right) = \frac{\pi}{6}$$



6. Determine whether each improper integral is convergent or divergent. If the integral converges, give the value of the integral. (10 pts. each)

(a) $\displaystyle\int_1^{\infty} \frac{1}{(4x-1)^2}\,dx$

Answer: This integral converges. Start by replacing ∞ by b and writing the integral as

$$\lim_{b\to\infty}\int_1^b \frac{1}{(4x-1)^2}\,dx$$

This integral is a u-sub with $u = 4x - 1$. Then $du = 4\,dx$ or $\frac{du}{4} = dx$. This transforms the integral into

$$\frac{1}{4}\int \frac{1}{u^2}\,du \;=\; \frac{1}{4}\int u^{-2}\,du \;=\; \frac{u^{-1}}{-4} \;=\; -\frac{1}{4(4x-1)}$$

Thus,

$$\lim_{b\to\infty}\int_1^b \frac{1}{(4x-1)^2}\,dx \;=\; \lim_{b\to\infty} -\frac{1}{4(4x-1)}\Bigg|_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{4(4b-1)} - \frac{-1}{12}\right) \;=\; 0 + \frac{1}{12} \;=\; \frac{1}{12}$$

(b) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{x+1}}\,dx$

Answer: This integral converges as well. Notice that the integrand is undefined when $x = -1$ so we replace -1 with b and write the integral as

$$\lim_{b\to -1^+}\int_b^1 \frac{1}{\sqrt{x+1}}\,dx$$

This integral is also a u-sub with $u = x + 1$. Then $du = 1\,dx$ and the integral transforms into

$$\int \frac{1}{\sqrt{u}}\,du \;=\; \int u^{-1/2}\,du \;=\; 2u^{1/2} \;=\; 2\sqrt{x+1}$$

Thus,

$$\lim_{b\to -1^+} \int_b^1 \frac{1}{\sqrt{x+1}}\,dx \;=\; \lim_{b\to -1^+} 2\sqrt{x+1}\,\Big|_b^1$$

$$= \lim_{b\to -1^+} (2\sqrt{2} - 2\sqrt{1+b}\,) \;=\; 2\sqrt{2} - 0 \;=\; 2\sqrt{2}$$