MATH 126-01 Calculus for the Social Sciences II

Exam #1 February 14, 2008 Prof. G. Roberts

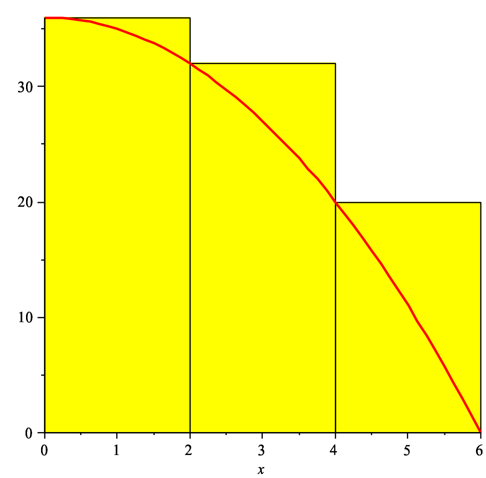
SOLUTIONS

1. Let $g(x) = 36 - x^2$ over the interval $0 \le x \le 6$.

   (a) Approximate the area under the graph of g from $0 \le x \le 6$ by using **three** rectangles and left endpoints (Left-hand Sum). (5 pts.)

   Answer: Since there are three rectangles whose total width covers a distance of 6, each rectangle has a width of 2. Therefore, LHS $= 2(g(0)+g(2)+g(4)) = 2(36+32+20) = 176$.

   (b) Sketch the graph of g along with the three rectangles used in the Left-hand Sum from part **(a)**. Is your sum an overestimate or underestimate? (5 pts.)



   Answer: Since the rectangles cover more area than what is under the curve, the Left-hand Sum is an overestimate.

   (c) Approximate the area under the graph of g by using **three** rectangles and midpoints (Midpoint Sum). (No sketch is required for this part.) (5 pts.)

   Answer: MS $= 2(g(1) + g(3) + g(5)) = 2(35 + 27 + 11) = 146$

   (d) Use calculus to compute the **exact** area under g from $0 \le x \le 6$. (6 pts.)

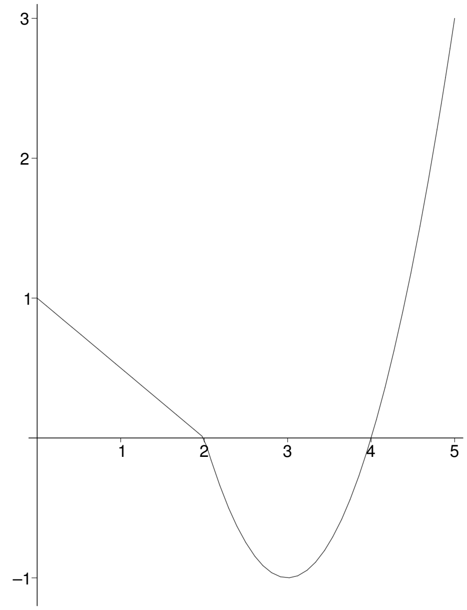
   Answer: By the Fundamental Theorem of Calculus (FTC), part II, the exact area can be found by computing:

$$\int_0^6 36 - x^2 \, dx = 36x - \frac{x^3}{3}\Big|_0^6 = 216 - 72 - 0 = 144$$

2. Define $F(x) = \displaystyle\int_0^x f(t)\, dt$ for $0 \le x \le 5$, where the graph of $f(t)$ is given below.

   (a) Find $F(0)$ and $F(2)$. (5 pts.)

   Answer: $F(0) = \int_0^0 f(t)\, dt = 0$ and $F(2) = \int_0^2 f(t)\, dt = \frac{1}{2} \cdot 2 \cdot 1 = 1$ (area of triangle).



(b) Find $F'(3)$ and $F''(1)$. (6 pts.)

Answer: Using FTC, part I, we have that $F'(x) = f(x)$ and also $F''(x) = f'(x)$. Therefore, $F'(3) = f(3) = -1$ (from the graph of f) and $F''(1) = f'(1) = -1/2$ since the slope of the graph of f at $x = 1$ is $-1/2$.

(c) On what interval(s) is F decreasing? (4 pts.)

Answer: Again, by FTC, part I, we have that $F'(x) = f(x)$. So F is decreasing when $F' < 0$ or when $f < 0$. The graph of f is below the horizontal axis ($f < 0$) when $2 < x < 4$.

(d) Over what interval(s) is F concave up? (4 pts.)

Answer: Using the fact that $F''(x) = f'(x)$ (from part **(b)**), we see that F is concave up when $F'' > 0$ or when $f' > 0$. From the graph of f, we see that f is increasing when $3 < x < 5$.

3. Evaluate each indefinite integral. (18 pts.)

a) $\displaystyle\int 3e^x - \frac{1}{\sqrt{x}}\, dx$

Answer: $\displaystyle\int 3e^x - \frac{1}{\sqrt{x}}\, dx = \int 3e^x\, dx - \int x^{-1/2}\, dx = 3e^x - 2x^{1/2} + c$

b) $\displaystyle\int x^2(1 - x^3)^4\, dx$

Answer: This is a u-sub with $u = 1 - x^3$. Let $u = 1 - x^3$, then $du = -3x^2\, dx$ or $(-1/3)du = x^2\, dx$. The integral is transformed into

$$\int u^4 \cdot -\frac{1}{3}\, du = -\frac{1}{3}\int u^4\, du = -\frac{1}{3}\cdot\frac{u^5}{5} + c = -\frac{1}{15}u^5 + c = -\frac{1}{15}(1 - x^3)^5 + c$$

c) $\displaystyle\int \frac{x+1}{x^2+2x+7}\,dx$

Answer: This is also a u-sub with $u = x^2+2x+7$. Then $du = 2x+2\,dx$ or $du/2 = x+1\,dx$. The integral is transformed into

$$\int \frac{1}{u}\cdot\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+2x+7| + c$$

4. Evaluate each definite integral. (21 pts.)

a) $\displaystyle\int_0^{\pi/2} e^{\cos\theta+1}\sin\theta\,d\theta$

Answer: This is a u-sub with $u = \cos\theta + 1$. Then $du = -\sin\theta\,d\theta$ or $-du = \sin\theta\,d\theta$. It is a bit easier to change the limits of integration as well. Thus, we have $\theta = 0 \Longrightarrow u = \cos(0)+1 = 2$ and $\theta = \pi/2 \Longrightarrow u = \cos(\pi/2)+1 = 1$. The integral is transformed into

$$\int_2^1 e^u \cdot -du = -\int_2^1 e^u\,du = \int_1^2 e^u\,du = e^u|_1^2 = e^2 - e = e(e-1)$$

b) $\displaystyle\int_0^1 \frac{2}{t^2+1}\,dt$

Answer: This integral is straight-forward if you remember the derivative of $\tan^{-1} t$. This problem is not a u-sub with $u = t^2+1$ because then $du = 2t\,dt$ but there is no t sitting in the integrand.

$$\int_0^1 \frac{2}{t^2+1}\,dt = 2\tan^{-1} t|_0^1 = 2\tan^{-1}(1) - 2\tan^{-1}(0) = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

Note that the answer is in radians because radians is a real, physical measurement (unlike degrees). An answer of 90 makes no sense because the area under the curve can be no bigger than 2.

c) $\displaystyle\int_{-3}^3 -2\sqrt{9-x^2}\,dx$

Answer: Although this looks like the integral of an odd function over a symmetric interval, its actually the area under a semi-circle of radius 3. We have

$$\int_{-3}^3 -2\sqrt{9-x^2}\,dx = -2\int_{-3}^3 \sqrt{9-x^2}\,dx = -2\cdot\frac{9\pi}{2} = -9\pi$$

using the formula for the area of a semi-circle $A = (1/2)\pi r^2$. To see that we are dealing with a circle of radius 3, let $y = \sqrt{9-x^2}$ and manipulate this expression into $x^2+y^2 = 3^2$ which is a circle centered at the origin of radius 3. The fact that it is a semi-circle (rather than a quarter-circle) comes from the limits of integration ranging from -3 to 3.

5. Some final conceptual questions: (21 pts.)

(a) Find the antiderivative $F(x)$ for $f(x) = \sec^2 x - e^{3x} + 2$ satisfying $F(0) = 1$.

Answer: We find the general antiderivative by integrating f. This gives $F(x) = \tan x - \frac{1}{3}e^{3x} + 2x + c$. The tricky part here is that

$$\int e^{3x}\, dx = \frac{1}{3}e^{3x},$$

a formula that can be checked by differentiation or derived using a u-sub with $u = 3x$. Then we solve $F(0) = 1$ for c, obtaining

$$1 = \tan(0) - \frac{1}{3}e^0 + 0 + c \implies c = \frac{4}{3}.$$

Thus, our final answer is

$$F(x) = \tan x - \frac{1}{3}e^{3x} + 2x + \frac{4}{3}.$$

(b) Given that $\displaystyle\int_2^5 p(t)\, dt = 3$ and $\displaystyle\int_{-1}^5 p(t)\, dt = 7$, find the value of $\displaystyle\int_{-1}^2 p(t)\, dt$.

Answer: Let $A = \displaystyle\int_{-1}^2 p(t)\, dt$. Using

$$\int_{-1}^2 p(t)\, dt \;+\; \int_2^5 p(t)\, dt \;=\; \int_{-1}^5 p(t)\, dt$$

we have the simple equation $A + 3 = 7$ so that our answer is $A = 4$.

(c) Find $\displaystyle\frac{d}{dx}\int_{\sin x}^3 e^{t^3}\, dt$.

Answer: This is FTC, part I along with the chain rule. We have

$$\frac{d}{dx}\int_{\sin x}^3 e^{t^3}\, dt = -\frac{d}{dx}\int_3^{\sin x} e^{t^3}\, dt = -e^{(\sin x)^3} \cdot \cos x$$

where the $\cos x$ comes from taking the derivative of the "inside" function $\sin x$ in the chain rule.