Calculus for the Social Sciences II, Spring 2008

The Beauty of Compound Interest

This worksheet is intended to introduce the very important number e through a particular real-world application, namely compound interest. You should round your answers to the nearest cent, but do not round off until the answer is obtained. Rounding off too soon can lead to numerical error.

When you invest in a mutual fund or deposit your money into a bank account or CD, you expect to make a certain rate of return r on your money. Your money can be **compounded** at different times during the year. Compounded yearly means that after one year, you make r percent of your initial investment and add it to your original. For example, if you deposit \$1,000 at an interest rate of 5%, then after one year you will have

$$1,000 + 0.05 \cdot 1,000 = 1.05 \cdot 1,000 = \$1,050.$$

If you had invested \$2,000 instead, you would have

$$2,000 + 0.05 \cdot 2,000 = 1.05 \cdot 2,000 = \$2,100$$

at the end of the year. The new amount is simply 1.05 times the previous amount. Notice that the change in the amount in your account (\$50 in the first case, \$100 in the second) is proportional to the amount you invest, with the proportionality constant equal to the interest rate of 5%. In other words, we have the following key fundamental relationship:

Change in value is proportional to amount invested

Continuing with our example, we will have

$$1.05 \cdot 1,050 = (1.05)^2 \cdot 1,000 = \$1,102.50$$

after two years and

$$1.05 \cdot 1,102.50 = (1.05)^2 \cdot 1,050 = (1.05)^3 \cdot 1,000 = \$1,157.63$$

after three years. A clear pattern has emerged. The amount of money in the account after t years is simply

$$A(t) \ = \ 1000(1.05)^t, \tag{1}$$

an *exponential* function! This is the magic of compound interest. Your money grows at an exponential rate, rather than linearly. If you only earned \$50 each year, you would have grown your account by \$150. However, because the money is compounded each year, your account has actually grown by \$157.63. This seems a small amount now, but it will have big consequences down the road.

If your money is compounded **quarterly**, the rate is adjusted to $r/4$, but your money is compounded 4 times a year (usually it is every 3 months). Compounded **monthly** means a rate of $r/12$ compounded 12 times a year while compounded **daily** means a rate of $r/365$ compounded every day of the year. In other words, the more your money is compounded, the lower the rate (seems bad), but the more often its compounded (seems good). So which is better, compounded yearly, quarterly, monthly, daily or possibly continuously (every possible instant)?

1. By generalizing equation (1), find the general formula for the amount of money $A(t)$ in an account if A_0 is invested at an interest rate r for t years.

2. Suppose you start with an initial investment of \$10,000 at a rate r of 8%. Find the amount $A(1)$ in your account after 1 year if it is compounded yearly versus if it is compounded quarterly. Find the amount in your account using each method after 10 years. Which method yields the greater value?

3. What is the general formula for $A(t)$ if A_0 is the initial quantity invested at an interest rate r after t years, if it is compounded quarterly?

4. Repeat the above two questions but compound the money in the account monthly and then daily. Which is the best after 10 years?

5. Compute the bases to 7 decimal places of the exponential functions in the above examples (yearly, quarterly, monthly and daily with interest rate $r = 8\%$). In other words, write each of the above functions in the standard form $A(t) = A_0\, b^t$. The base is the value of b. Compare the bases to each other. Which is the largest?

6. In theory we could compound an account every minute, every second or even every nanosecond. What is the general formula for the amount of an investment $A(t)$ if the initial amount is A_0, the interest rate is r and the account is compounded m times a year? Assume t is in years.

   There are important connections between compound interest, differential equations and the special number e. The term used for compounding an investment at every possible instant over the course of a year is **compounded continuously**. Picture a very fast mathematician calculating your account over and over again without ever taking a break. The way to derive the formula for compounding continuously is to rewrite our key concept "Change in value is proportional to amount invested" in the form of the differential equation

   $$\frac{dA}{dt} = rA$$

   where r is the interest rate. Luckily we can solve this ODE. If $A(0) = A_0$ is the initial amount invested, solve the ODE to find a formula for $A(t)$.

7. Using the formula derived in the previous question, how much money is in an account where \$10,000 is invested at a rate of 8% compounded continuously for 10 years? What is the base b in this case? Compare to your previous answers. Which method of compounding yields the most money?

8. Based on your work thus far, explain why the special number e is defined as the following:

   $$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e.$$

   Hint: Try setting $r = 1$ and comparing the formula for interest compounded m times a year from question #6 with the formula for compounding interest continuously from question #7. What are the bases in each case?