MATH 125-03/04 Calculus for the Social Sciences I

Exam #3 Solutions **December 3, 2009** **Prof. G. Roberts**

1. Calculate the derivative of each function. Be sure to **simplify** your answer. (36 pts.)

(a) $f(x) = x^3 + 3^x + e^3$

Answer: Note that e^3 is just a constant. Thus, $f'(x) = 3x^2 + \ln 3 \cdot 3^x$

(b) $g(t) = \dfrac{5t^2 - 1}{3t^2 + 2}$

Answer:

$$\begin{aligned} g'(t) &= \frac{(3t^2+2)(10t) - (5t^2-1)(6t)}{(3t^2+2)^2} \\ &= \frac{30t^3 + 20t - 30t^3 + 6t}{(3t^2+2)^2} \\ &= \frac{26t}{(3t^2+2)^2} \end{aligned}$$

(c) $h(x) = \sqrt{\tan(x^2+1)}$

Answer: First write $h(x) = (\tan(x^2+1))^{1/2}$. Applying the chain rule twice, we have

$$h'(x) = \frac{1}{2}(\tan(x^2+1))^{-1/2} \cdot \sec^2(x^2+1) \cdot 2x = \frac{x\sec^2(x^2+1)}{\sqrt{\tan(x^2+1)}}$$

(d) $y = e^{-4x}\sin(2x)$

Answer: Using the product rule and then the chain rule to differentiate each function, we have

$$\frac{dy}{dx} = -4e^{-4x} \cdot \sin(2x) + e^{-4x} \cdot \cos(2x) \cdot 2 = 2e^{-4x}(\cos(2x) - 2\sin(2x))$$

(e) $g(\theta) = \ln(\cos\theta)$

Answer:

$$g'(\theta) = \frac{1}{\cos(\theta)} \cdot -\sin(\theta) = -\tan(\theta)$$

(f) $h(t) = \tan^{-1}(\sqrt{t}\,)$ **Answer:**

$$h'(t) = \frac{1}{1+(\sqrt{t})^2} \cdot \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}\,(1+t)}$$

2. For the equation below, use implicit differentiation to calculate dy/dx. (12 pts.)

$$x^3 \sin y + \cos(2y) = e^{y^3} + 7x^2$$

Answer: Differentiating each side with respect to x and treating $y = y(x)$ as a function of x, we have

$$3x^2 \sin y + x^3 \cos y \cdot \frac{dy}{dx} - \sin(2y) \cdot 2\frac{dy}{dx} \;=\; e^{y^3} \cdot 3y^2 \frac{dy}{dx} + 14x.$$

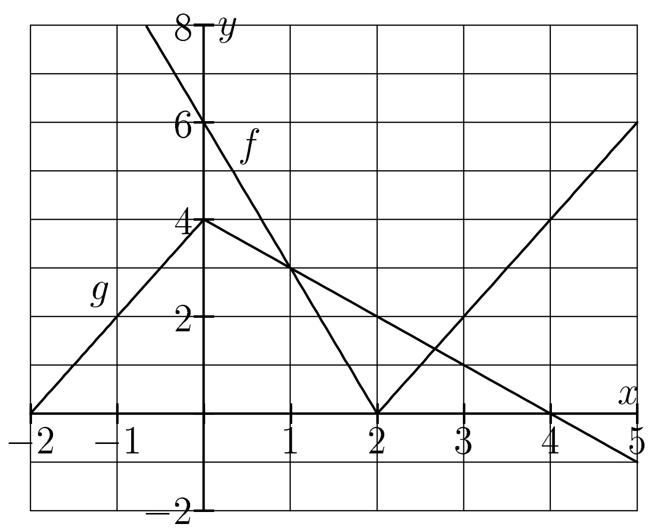
Grouping all terms with dy/dx together on one side of the equation yields

$$x^3 \cos y \cdot \frac{dy}{dx} - 2\sin(2y) \cdot \frac{dy}{dx} - 3y^2 e^{y^3} \cdot \frac{dy}{dx} \;=\; 14x - 3x^2 \sin y$$

which gives, after factoring out the dy/dx on the left-hand side,

$$\frac{dy}{dx} \;=\; \frac{14x - 3x^2 \sin y}{x^3 \cos y - 2\sin(2y) - 3y^2 e^{y^3}}.$$

3. The graphs of the functions $f(x)$ and $g(x)$ are given below. Suppose that $h(x) = f(x)/g(x)$ and $C(x) = f(g(x))$. If they exist, find both $h'(3)$ and $C'(3)$. (14 pts.)



Answer: $h'(3) = 4$ and $C'(3) = 3$

To find $h'(3)$, use the quotient rule and fill in the appropriate values according to the graph (derivative = slope). We have that $h'(x) = (g(x)f'(x) - f(x)g'(x))/(g(x))^2$ so that

$$\begin{aligned} h'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} \\ &= \frac{1 \cdot 2 - 2 \cdot (-1)}{1^2} \\ &= 4. \end{aligned}$$

To find $C'(3)$, use the chain rule. We have that $C'(x) = f'(g(x)) \cdot g'(x)$ so that

$$C'(3) \;=\; f'(g(3)) \cdot g'(3) \;=\; f'(1) \cdot g'(3) \;=\; -3 \cdot -1 \;=\; 3.$$

4. Suppose that a particle is traveling along the x-axis with position given by

$$s(t) = t^3 - 6t^2 - 15t + 20,$$

where s is measured in feet and t in seconds. Assume that $t \geq 0$. (13 pts.)

(a) Find the velocity and acceleration functions.

Answer: $v(t) = s'(t) = 3t^2 - 12t - 15$ and $a(t) = s''(t) = 6t - 12$.

(b) When is the particle moving to the right? When is it moving to the left?

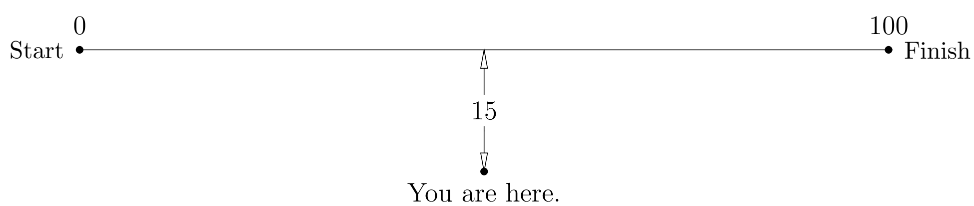
Answer: Note that $v(t) = 3(t^2 - 4t - 5) = 3(t-5)(t+1)$. The particle is traveling to the right when $s(t)$ is increasing or when $v(t) > 0$. Using a number line, this occurs when $t > 5$. The particle is traveling to the left when $s(t)$ is decreasing or when $v(t) < 0$. This occurs for $0 \leq t < 5$.

(c) Find the total distance traveled by the particle during the first 10 seconds.

Answer: 450 feet.

Since the particle changes direction at $t = 5$, we must calculate the distance traveled in the first 5 seconds and then add that quantity to the distance traveled over $5 \leq t \leq 10$. We compute that $s(0) = 20$ and $s(5) = -80$. Thus, the particle starts out traveling left a distance of $20 - (-80) = 100$ ft. Then, since $s(10) = 270$, the particle travels $270 - (-80) = 350$ to the right. Combining these together gives a total distance of 450 feet.

5. You are at a Holy Cross track meet watching the 100 meter dash, sitting in the bleachers 15 meters back from the midway point of the race. Your friend Natasha (a.k.a., the speedster) is running down the track at a constant speed of 9 meters per second. How fast is the distance between you and your friend changing when she is 70 meters into the race? (13 pts.)



Answer: 7.2 meters/sec

Draw a right triangle with one leg equal to 15, the other leg equal to x and the hypotenuse equal to y. The right angle of this triangle occurs at the midway point of the race, 50 meters along the track. The hypotenuse y is the distance between you and Natasha. Both the quantities x and y are changing over time but the distance 15 is fixed because you are not moving. When Natasha is 70 meters into the race, we have $x = 20$. We want to find dy/dt when $x = 20$.

By the Pythagorean Theorem, we have $15^2 + x^2 = y^2$. Differentiating this equation with respect to t yields $2x \cdot \frac{dx}{dt} = 2y \cdot \frac{dy}{dt}$ which simplifies to

$$\frac{dy}{dt} = \frac{x}{y} \cdot \frac{dx}{dt}.$$

We are given that $dx/dt = 9$. To find y when $x = 20$, use the Pythagorean Theorem or recognize the Pythagorean triple $(15, 20, 25) = 5 \cdot (3, 4, 5)$. Thus, $y = 25$ and we have

$$\frac{dy}{dt} = \frac{20}{25} \cdot 9 = \frac{36}{5} = 7.2 \text{ meters/sec.}$$

6. Find the absolute maximum and absolute minimum values of the function $f(x) = \frac{1}{2}x - \ln(2x)$ over the interval $[\frac{1}{2}, 6]$. (12 pts.)

Answer:

First find any critical points of f on the interval. We have

$$f'(x) = \frac{1}{2} - \frac{1}{2x} \cdot 2 = \frac{1}{2} - \frac{1}{x}.$$

Solving $f'(x) = 0$ for x gives $1/2 = 1/x$ or $x = 2$. Next, compute the values of f at the endpoints of the given interval. We have $f(1/2) = 1/4 - \ln 1 = 1/4$ and $f(6) = 3 - \ln(12) \approx 0.515$. Since $f(2) = 1 - \ln 4 \approx -0.386$, we find (comparing the three function values) that f has an absolute maximum of $3 - \ln(12)$ at $x = 6$ and an absolute minimum of $1 - \ln 4$ at $x = 2$.