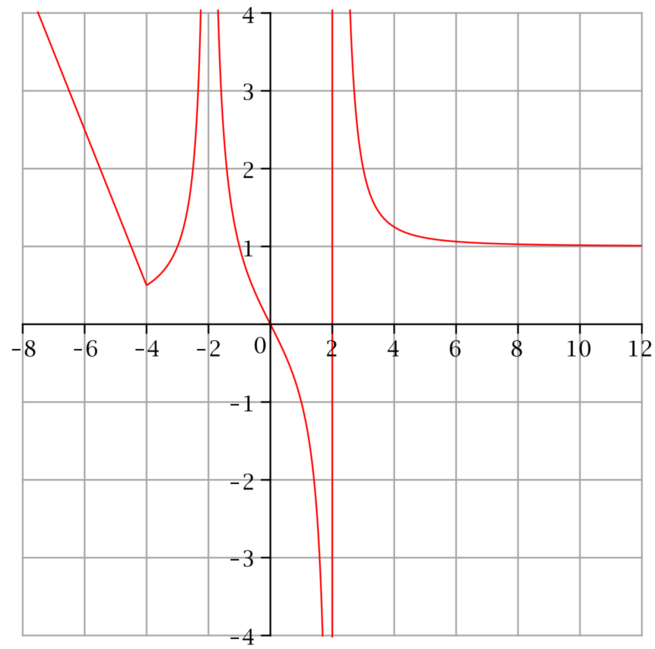
MATH 125-03/04 Calculus for the Social Sciences I

Exam #2 SOLUTIONS **October 29, 2009** **Prof. G. Roberts**

1. The graph of $f(x)$ is shown below. Use it to answer each of the following questions. Note that ∞ or $-\infty$ are acceptable answers for the limit problems. (18 pts.)



(a) Evaluate $\lim_{x \to 2^-} f(x)$

Answer: $-\infty$

(b) Evaluate $\lim_{x \to 2} f(x)$

Answer: Does not exist. The left and right-hand limits are different.

(c) Evaluate $\lim_{x \to -2} f(x)$

Answer: ∞

(d) Evaluate $\lim_{x \to \infty} f(x)$

Answer: 1. The graph has a horizontal asymptote at $y = 1$.

(e) List any x-values where f is **not** continuous.

Answer: $x = -2, 2$

(f) List any x-values where f is **not** differentiable.

Answer: $x = -4, -2, 2$. Note that any places where the function is not continuous will also be places the function is not differentiable.

2. Evaluate each of the following limits, if they exist. Note that ∞ or $-\infty$ are acceptable answers. Be sure to show your work. (6 pts. each)

(a) Evaluate $\lim_{x\to\pi}(\cos(x)+\sin(x))$

Answer: The function $f(x) = \cos(x) + \sin(x)$ is continuous, so to evaluate this limit, compute $f(\pi) = \cos(\pi) + \sin(\pi) = -1 + 0 = -1$.

(b) Evaluate $\lim_{x\to 5}\dfrac{x^2-3x-10}{x^2-25}$

Answer: $\frac{7}{10}$

$$\lim_{x\to 5}\frac{x^2-3x-10}{x^2-25} = \lim_{x\to 5}\frac{(x-5)(x+2)}{(x-5)(x+5)} = \lim_{x\to 5}\frac{x+2}{x+5} = \frac{7}{10}.$$

(c) Evaluate $\lim_{x\to\infty}\dfrac{2x^3-3x^2+4}{5x^3+7x-16}$

Answer: $\frac{2}{5}$. Divide the top and bottom of the fraction by the highest power x^3. This gives

$$\lim_{x\to\infty}\frac{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{4}{x^3}}{\frac{5x^3}{x^3}+\frac{7x}{x^3}-\frac{16}{x^3}} = \frac{\lim_{x\to\infty} 2-\frac{3}{x}+\frac{4}{x^3}}{\lim_{x\to\infty} 5+\frac{7}{x^2}-\frac{16}{x^3}} = \frac{2-0+0}{5+0-0} = \frac{2}{5}.$$

3. **(a)** State a limit definition for the derivative of a function $f(x)$ at the point $x = a$. (5 pts.)

Answer:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

(b) Use your limit definition from part **(a)** to find $f'(8)$ where $f(x) = \sqrt{2x}$. (10 pts.)

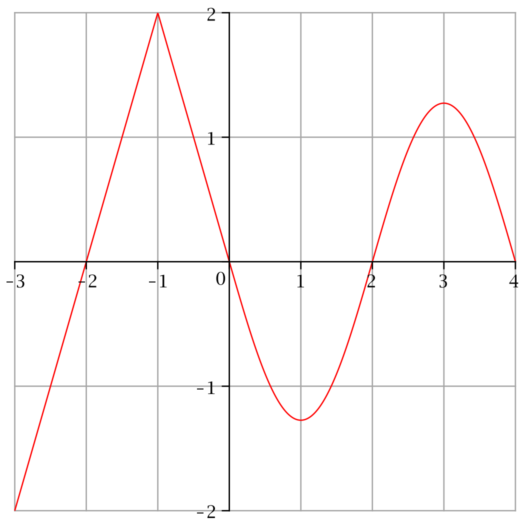
Answer: $f'(8) = 1/4$.

$$\begin{aligned}
f'(8) &= \lim_{h\to 0}\frac{f(8+h)-f(8)}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2(8+h)}-4}{h} \\
&= \lim_{h\to 0}\frac{(\sqrt{2(8+h)}-4)(\sqrt{2(8+h)}+4)}{h(\sqrt{2(8+h)}+4)} \\
&= \lim_{h\to 0}\frac{2(8+h)-16}{h(\sqrt{2(8+h)}+4)} \\
&= \lim_{h\to 0}\frac{2h}{h(\sqrt{2(8+h)}+4)} \\
&= \lim_{h\to 0}\frac{2}{\sqrt{2(8+h)}+4} \\
&= \frac{2}{4+4} = \frac{2}{8} = \frac{1}{4}.
\end{aligned}$$

(c) Find the equation of the tangent line to $f(x) = \sqrt{2x}$ at the point $x = 8$. (5 pts.)

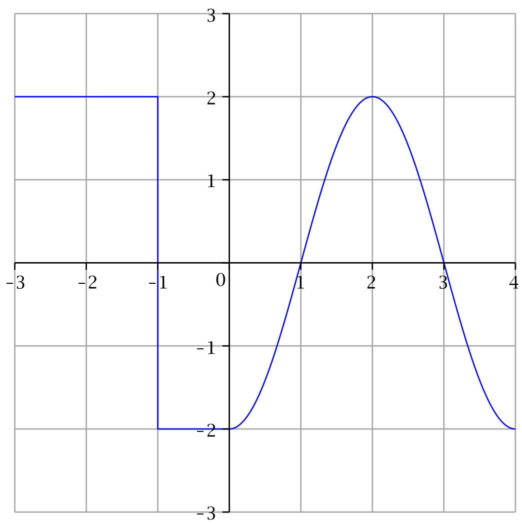
Answer: $y = \frac{1}{4}x + 2$. From part **(b)**, we have that $m = f'(8) = 1/4$. Thus, $y = \frac{1}{4}x + b$. To find b, we use the point $(8, 4)$ since $x = 8$ implies $y = f(8) = 4$. Therefore, we have $4 = \frac{1}{4} \cdot 8 + b$ or $4 = 2 + b$ which implies $b = 2$. Thus the equation of the tangent line is $y = \frac{1}{4}x + 2$.

4. Given the graph of the function $g(x)$ below, sketch the graph of the derivative $g'(x)$ on the axes provided. (12 pts.)



Answer:

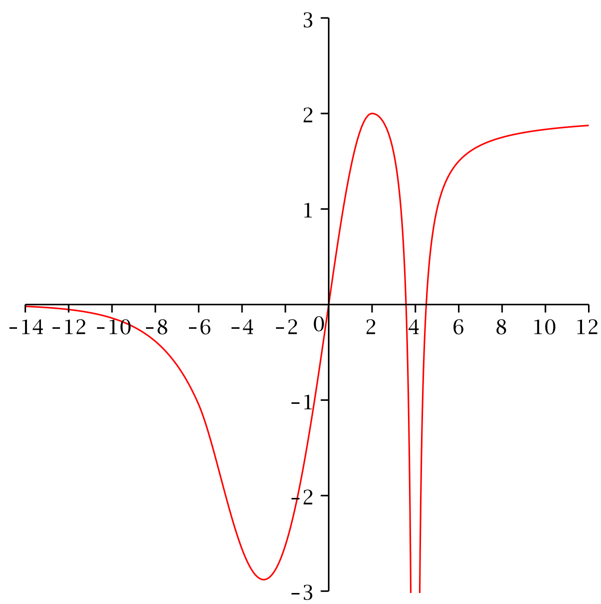
Note that the graph of the derivative will have a hole at $x = -1$ as $g'(-1)$ does not exist (corner).



5. Sketch the graph of a function $f(x)$ satisfying all of the following properties: (12 pts.)

- f is continuous at all x except for $x = 4$
- $f(0) = 0$ and $f(4)$ does not exist
- $f'(-3) = 0$ and $f'(2) = 0$
- $\lim_{x\to 4} f(x) = -\infty$
- $\lim_{x\to \infty} f(x) = 2$
- $\lim_{x\to -\infty} f(x) = 0$
- $f'(x) < 0$ for $x < -3$ or $2 < x < 4$
- $f'(x) > 0$ for $-3 < x < 2$ or $x > 4$
- $f''(x) < 0$ for $x < -5$ or $0 < x < 4$ or $x > 4$
- $f''(x) > 0$ for $-5 < x < 0$

Answer:



6. Some final conceptual questions. You must show your work to receive any partial credit. (20 pts.)

(a) If $6x - 4 \leq h(x) \leq x^2 + 5$ for all x, find $\lim_{x\to 3} h(x)$.

Answer: Using the Squeeze Theorem, since $\lim_{x\to 3} 6x - 4 = 14$ and $\lim_{x\to 3} x^2 + 5 = 14$, we have that $\lim_{x\to 3} h(x) = 14$.

(b) Suppose that $P(s)$ represents the profit earned in dollars for selling s stereos. Which of the following best describes the meaning of $P'(500) = 100$?

(i) The profit earned from selling 100 stereos is \$500.
(ii) The profit earned from selling 500 stereos is \$100.
(iii) Selling the 501st stereo will earn, approximately, an additional \$100 in profit.
(iv) Selling the 101st stereo will earn, approximately, an additional \$500 in profit.
(v) The rate of change of the profit is \$500 per stereo after selling 100 stereos.

Answer: (iii)

(c) A stock analyst gives the following advice: "Although the value $v(t)$ of a share in Owen Inc. continues to decline, it is declining at a slower rate. It might be wise **not** to completely sell off the stock." Interpret this statement in terms of the signs (positive, negative or zero) of $v'(t)$ and $v''(t)$, where $v(t)$ represents the value of the stock at time t.

Answer: The first derivative $v'(t)$ is negative because the value of the stock is decreasing. However, since it is declining at a slower rate, the second derivative $v''(t)$ is positive ($v(t)$ is concave up).

(d) Find and simplify $k'(x)$ if $k(x) = \dfrac{3}{x^5} + 6x^{2/3} + e^{\pi}$.

Answer: First, rewrite $k(x)$ as $k(x) = 3x^{-5} + 6x^{2/3} + e^{\pi}$. Using the power rule, we have $k'(x) = -15x^{-6} + \frac{2}{3} \cdot 6x^{-1/3} + 0$ (note that e^{π} is just a constant.) Simplifying, we have $k(x) = -15x^{-6} + 4x^{-1/3}$.