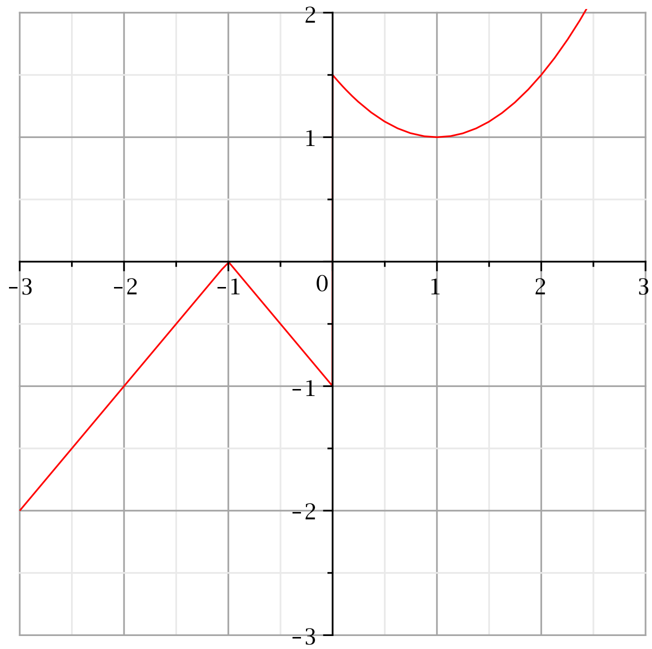
MATH 125-03/04 Calculus for the Social Sciences I

Exam #1 SOLUTIONS **October 1, 2009** **Prof. G. Roberts**

1. The **ENTIRE** graph of $f(x)$ is shown below. Use it to answer each of the following questions: (26 pts.)



(a) What is the domain of f? $[-3, 2.4]$

(b) What is the range of f? $[-2, 0] \cup [1, 2]$. Note that there is a gap in the function and that no y-values between 0 and 1 have pre-images in the domain.

(c) Is f a one-to-one function? No, it fails the horizontal line test. Specifically, there are output values of the function that have more than one pre-image in the domain. For example, $f(0) = f(2) = 1.5$.

(d) For what value(s) of x does $f(x) = 1.5$? $x = 0$ and $x = 2$

(e) What is $(f \circ f)(-2)$? 0.
$f(f(-2)) = f(-1) = 0$.

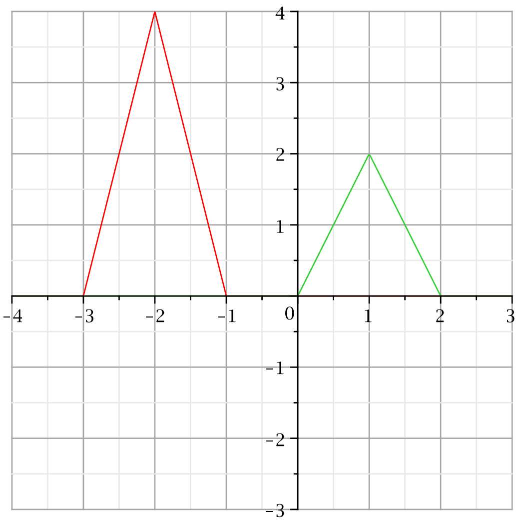
(f) Evaluate $\lim_{x \to 0^+} f(x) \; = \; 1.5$

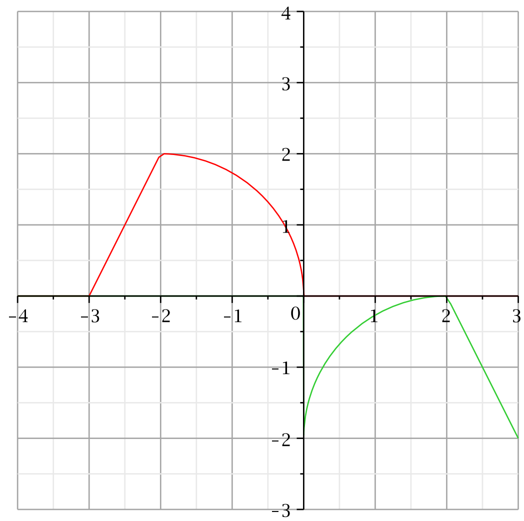
(g) Evaluate $\lim_{x \to 0^-} f(x) \; = \; -1$

(h) Evaluate $\lim_{x \to 0} f(x)$ does not exist.

2. For each of the graphs shown below, one or more transformations have been applied to the original function $f(x)$ to obtain a new function $g(x)$. In mathematical terms, state the formula for $g(x)$ in terms of $f(x)$. For example, a typical answer might be $g(x) = 3f(2x+1)$. (12 pts.)

(a) $g(x) = \underline{\frac{1}{2}f(x-3) \text{ or } \frac{1}{2}f(-x-1)}$ **(b)** $g(x) = \underline{f(-x) - 2}$





For part **(a)**, we shift the graph of $f(x)$ right three units ($f(x-3)$) and compress it by a factor of 2, giving $\frac{1}{2}f(x-3)$. Alternatively, we could reflect the graph about the y-axis ($f(-x)$), shift left by one unit ($f(-(x+1))$ and then compress it by a factor of 2, giving $\frac{1}{2}f(-x-1)$. For part **(b)**, we reflect the graph about the y-axis ($f(-x)$) and shift it down by two units, giving $f(-x)-2$.

3. A population of mold has begun to develop on a piece of bread you brought back to your dormitory from Kimball dining hall. Initially, the population is only 100 cells, but after two hours, it has **tripled** to 300 cells. Assume the cells continue to grow exponentially at this rate. Use the variables t for time and P for population of mold cells.

   (a) How many mold cells are on the bread after six hours? (4 pts.)

   Answer: 2700. After two hours there are three times as much as 100, or 300. After another two hours (four total) there will be three times as much as 300 or 900 cells. After another two hours (six total), there will be three times as much again, or 2700 mold cells.

   (b) Find an expression in the form $P(t) = c \cdot a^t$ modeling the number of mold cells on the bread after t hours. (5 pts.)

   Answer: We have $c = P_0 = 100$. This can be found by plugging the point $t = 0, P = 100$ into $P(t) = c\,a^t$. Then, using $t = 2, P = 300$ as a second data point, we have $300 = 100 \cdot a^2$. This reduces to $a^2 = 3$ or $a = \sqrt{3}$. Thus the formula for the number of cells after t hours is $P(t) = 100(\sqrt{3})^t = 100 \cdot 3^{t/2}$.

   (c) Public Safety will confiscate your bread if the number of mold cells reaches one million. After how many hours will this happen? Give the **exact** answer as well as an approximation rounded to two decimal places. (6 pts.)

   Answer: Using the expression derived in part **(b)**, we must solve the equation $1,000,000 = 100(\sqrt{3})^t$ for t. The first step is to divide both sides by 100. This gives $10,000 = (\sqrt{3})^t$. To solve this we take the natural logarithm of both sides, $\ln(10,000) = \ln(\sqrt{3}^{\,t}) = t \cdot \ln(\sqrt{3})$. Dividing both sides by $\ln(\sqrt{3})$ gives

$$t = \frac{\ln(10,000)}{\ln(\sqrt{3})} \quad \text{or} \quad t = \frac{2\ln(10,000)}{\ln 3}.$$

Using a calculator and rounding to the nearest hundredth, we find that $t \approx 16.77$ hours.

4. The table below shows the entries for a linear function, an exponential function and a trigonometric function.

x	$f(x)$	$g(x)$	$h(x)$
-1	18	0	10
0	6	6	6
1	2	0	2
2	2/3	-6	-2
3	2/9	0	-6

(a) Match each of the functions f, g or h with its correct type (linear, exponential or trigonometric). **Explain your answer briefly.** (6 pts.)

Answer: f is an exponential function because there is a constant ratio of 1/3 between successive terms. g is a trigonometric function because the values oscillate and appear to be the values of a scaled cosine function (specifically $6\cos(\frac{\pi}{2}x)$). h is a linear function because the difference between successive values is constant at -4.

(b) Find a formula for the linear function. (5 pts.)

Answer: Using the points $(-1, 10)$ and $(0, 6)$, we find the slope is $(10-6)/(-1-0) = -4$ and the y-intercept is 6. Therefore, $h(x) = -4x + 6$.

(c) Find a formula for the exponential function. (5 pts.)

Answer: Using the points $(0, 6)$ and $(1, 2)$ and the formula $f(x) = c \cdot a^x$, we have the equations $6 = c \cdot a^0$ and $2 = c \cdot a^1$. These simplify to $c = 6$ and $ac = 2$. Substituting $c = 6$ into the second equation gives $6a = 2$ which implies $a = 1/3$. Therefore, $f(x) = 6(1/3)^x$.

(d) Fill in the bottom row of the table. (6 pts.)

Answer: This can be done by continuing the patterns for each function or using the formulae derived in the previous two parts. We have $f(3) = 6(1/3)^3 = 6/27 = 2/9$ and $h(3) = -4(3) + 6 = -6$. $g(3) = 0$ because the trig function is periodic.

5. Some final conceptual questions. You must show your work to receive any partial credit. (25 pts.)

(a) The function $g(x) = \cos x + x^2 + 3$ is <u>even</u>.

The three functions $\cos x, x^2$ and 3 are all even functions. Since the sum of even functions remains even, g is an even function. One could also check that $g(-x) = g(x)$ which is the definition of an even function.

(b) Simplify $\log_2 32 + e^{\ln 15}$

Answer: 20. We have $\log_2 32 = 5$ since $2^5 = 32$. We also have $e^{\ln 15} = 15$ since $e^{\ln x} = x$ for any x (definition of $\ln x$.) Therefore, the sum of these two numbers is 20.

(c) Find the domain and range of the function $g(x) = \ln(x - 3)$.

Answer: The domain of $\ln x$ is $x > 0$ and the range is $\mathbb{R}$. (Draw a graph). Since the graph of $\ln(x - 3)$ is found by shifting the graph of $\ln x$ right by 3 units, we must shift the domain right by three units as well. The range is un-effected. Thus, the domain is $\{x \in \mathbb{R} : x > 3\}$ or $(3, \infty)$ and the range is all real numbers or $\mathbb{R}$ or $(-\infty, \infty)$.

(d) Find $\lim_{x \to 2} \dfrac{x^2 - 4}{2x^2 - 3x - 2}$

Answer: 4/5. Although we could do this limit using a calculator and plugging in values very close to 2 (such as 1.9999 and 2.00001), the easiest way is to factor the top and bottom and simplify. We have

$$\lim_{x \to 2} \frac{x^2 - 4}{2x^2 - 3x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(2x+1)(x-2)} = \lim_{x \to 2} \frac{x+2}{2x+1} = \frac{4}{5}.$$

The cancellation of $x - 2$ is justified because x never actually reaches 2 so that we are not dividing by zero.

(e) If $s(t) = t^2 + 3t$ represents the distance in feet a ball has traveled after t seconds, then the average velocity over the interval $[2, 4]$ is ____________________________.

Answer: 9 ft/sec. The average velocity over the interval $[2, 4]$ is

$$\frac{s(4) - s(2)}{4 - 2} = \frac{28 - 10}{2} = 9.$$