MATH 125, Fall 2009

Worksheet on Optimization Problems (Section 4.6)

Some comments:

- Word problems are hard! They are hard for everyone — students, grad students, professors, authors and engineers. Its fine to get discouraged or frustrated. This is to be expected. But they are also really important! Remember that calculus is essentially an applied subject and that problem solving is what people do in the "real world." No one is going to offer you a job because you can take the derivative of a function. People like to hire good problem solvers and that's where getting proficient at doing word problems really pays off.

- It's all about the set up! Draw a picture and label variables. The eventual goal is to arrive at a function of one variable representing a quantity to be optimized. For example, if you are finding the smallest surface area S, then you want to find an equation for S as a function of one variable. So a formula like $S = 2w^2 + 4wl$ needs to be reduced to a formula with just w or l on the right-hand side. Usually there will be some condition given that will allow you to substitute in for one variable to accomplish this reduction. Once you have a function of one variable, use your Calculus techniques to find either the max or min.

- **Applications to Business and Economics:** Here are some important functions to understand for problems in economics. The **cost function** $C(x)$ represents the cost of producing x units of a certain item. The **marginal cost** function is simply $C'(x)$, the rate of change of the cost with respect to x. The **demand** or **price function** is $p(x)$, the price per unit a company will charge if it is selling x units. This is typically a decreasing function, since the more a company sells, the lower the price per unit it can charge. Alternatively, the lower the price, the more of x consumers are likely to buy.

  Since $p(x)$ represents the price **per unit**, the total amount of revenue earned for selling x units of a particular item is given by the **revenue function**

  $$R(x) \;=\; x \cdot p(x).$$

  The **marginal revenue** function is then $R'(x)$. The amount of profit earned is given by the **profit function**

  $$P(x) \;=\; R(x) - C(x)$$

  which is simply the revenue earned in selling x units minus the cost to produce those x units. Finally, the **marginal profit** function is $P'(x)$ which measures the rate of change of profit with respect to x. If this is positive, then you will continue to make money by selling more goods, so you should continue to sell. If this is negative, you are losing money by selling more goods (cost outweighs the profit earned) and you should stop selling more goods. Thus, profit is maximized at the x-value where $P'(x) = 0$. This is precisely how we find the maximum of a function — search for its critical points.

Sample Problems:

1. Find two positive numbers whose product is 100 and whose sum is a minimum.

   Answer: Start by calling the two numbers x and y. You want to minimize the quantity $S = x + y$. Before you can do this you need to write S as a function of one variable. Find a relationship between x and y and then use this to substitute into the right-hand side of S to get a function of one variable. Then find the minimum of this function and solve the problem. Be sure to check that your solution really is an absolute minimum.

2. **(Shape of a Can)** An aluminum can needs to be designed to hold 100 cm^3 of juice. The can is cylindrical with flat caps at both ends. What are the dimensions of the can which use the least amount of material? **Hint:** The volume of a cylinder is $V = \pi r^2 h$. Try to minimize the surface area (including the two ends).

3. **(Maximizing Profit)** Suppose that $C(x) = 300 + 1.1x$ and $p(x) = 5 - 0.003x$ represent the cost and demand functions, respectively, for a certain company. Find the production level x that maximizes profit.

4. **(A Necklace Business)** During the summer, Tim makes and sells necklaces on the beach. Last summer he sold the necklaces for \$10 each and his sales averaged 20 per day. When he increased the price by \$1, he found that he lost two sales per day.

   (a) Find the demand function assuming that it is linear.

   (b) If the material for each necklace costs Tim \$6, what should the selling price be to maximize profit?

5. **(Landscape Architecture)** A landscape architect named Tina plans to enclose a 3500 square foot rectangular region in a botanical garden. She will use shrubs costing \$25 per foot along three sides and fencing costing \$10 per foot along the fourth side. The owners want to spend as little as possible on the construction. What is the minimum total cost of building the garden?

6. **(Maximizing Area)** A rectangle has one side on the x-axis and two vertices on the curve $y = \dfrac{1}{1+x^2}$. Find the vertices of the rectangle with maximum area.

   Answer: First, use your curve sketching techniques to sketch the graph of the function. Notice that y is an **even** function. Then draw a rectangle with the base on the x-axis whose upper vertices are on the curve. What symmetry do you notice about your rectangle? Label the lower right vertex $(x, 0)$ and find the area $A(x)$ of the rectangle as a function of x. Find where A has a maximum and finish the problem.