MATH 125 Calculus for the Social Sciences I

Exam #3 SOLUTIONS **Prof. G. Roberts**

1. Calculate the derivative of each function. Be sure to simplify your answer. (32 pts.)

(a) $f(x) = x^2 + 2^x + e$

Answer: $f'(x) = 2x + \ln 2 \cdot 2^x$

(b) $g(t) = \dfrac{5t^2 - 1}{3t^2 + 2}$

Answer:

$$\begin{aligned} g'(t) &= \frac{(3t^2+2)(10t) - (5t^2-1)(6t)}{(3t^2+2)^2} \\ &= \frac{30t^3 + 20t - 30t^3 + 6t}{(3t^2+2)^2} \\ &= \frac{26t}{(3t^2+2)^2} \end{aligned}$$

(c) $h(x) = \sqrt{\tan(x^2+1)}$

Answer: First write $h(x) = (\tan(x^2+1))^{1/2}$. Then, applying the chain rule twice, we have

$$\begin{aligned} h'(x) &= \frac{1}{2}\left(\tan(x^2+1)\right)^{-1/2} \cdot \sec^2(x^2+1) \cdot 2x \\ &= \frac{x \sec^2(x^2+1)}{\sqrt{\tan(x^2+1)}} \end{aligned}$$

(d) $y = e^{-2x}\sin(3x)$

Answer: This problem requires the product rule first, then the chain rule to differentiate each function.

$$\begin{aligned} dy/dx &= -2e^{-2x} \cdot \sin(3x) + e^{-2x} \cdot \cos(3x) \cdot 3 \\ &= e^{-2x}\left(-2\sin(3x) + 3\cos(3x)\right) \end{aligned}$$

(e) $g(\theta) = \ln(\cos\theta)$

Answer: Use the chain rule.

$$g'(\theta) = \frac{1}{\cos\theta} \cdot -\sin\theta = -\tan\theta$$

(f) $h(t) = \sin^{-1}(\sqrt{t}\,)$

Answer: Use the chain rule.

$$h'(t) = \frac{1}{\sqrt{1-(\sqrt{t})^2}} \cdot \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{1-t}\,\sqrt{t}} = \frac{1}{2\sqrt{t-t^2}}$$

2. Suppose that a curve is described by the equation

$$x^3 \sin y + \cos(2y) = y^3 + x\ .$$

(a) Use implicit differentiation to calculate dy/dx. (12 pts.)

Answer: Thinking of $y = y(x)$, we differentiate both sides of the equation with respect to x to obtain

$$3x^2 \sin y + x^3 \cos y \cdot \frac{dy}{dx} - \sin(2y) \cdot 2\frac{dy}{dx} = 3y^2\frac{dy}{dx} + 1$$

Then group all terms with $\frac{dy}{dx}$ in them:

$$x^3 \cos y \frac{dy}{dx} - 2\sin(2y)\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 1 - 3x^2 \sin y$$

Factor out $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(x^3 \cos y - 2\sin(2y) - 3y^2\right) = 1 - 3x^2 \sin y$$

and then divide to obtain

$$\frac{dy}{dx} = \frac{1 - 3x^2 \sin y}{x^3 \cos y - 2\sin(2y) - 3y^2}\ .$$

(b) Find the equation of the tangent line to the curve at the point $(1,0)$. (6 pts.)

Answer: The slope of the tangent line is given by the value of $\frac{dy}{dx}$ at the point $(1,0)$. Plugging in $x = 1$ and $y = 0$ into our answer from part **(a)** gives

$$m = \frac{dy}{dx}\Big|_{(1,0)} = \frac{1 - 3\cdot 1\cdot 0}{1\cdot 1 - 2\cdot 0 - 0} = 1$$

Then, using $y = x + b$, we plug in $x = 1$ and $y = 0$ to obtain $0 = 1 + b$ or $b = -1$. Thus, the equation of the tangent line at $(1,0)$ is

$$y = x - 1.$$

3. Suppose that $C(x) = 2500 + 5x - 0.04x^2 + 0.0003x^3$ represents the cost of producing x number of plasma television sets.

(a) Find the marginal cost function. (5 pts.)

Answer:

The marginal cost function is defined as the derivative of the cost function: $C'(x) = 5 - 0.08x + 0.0009x^2$.

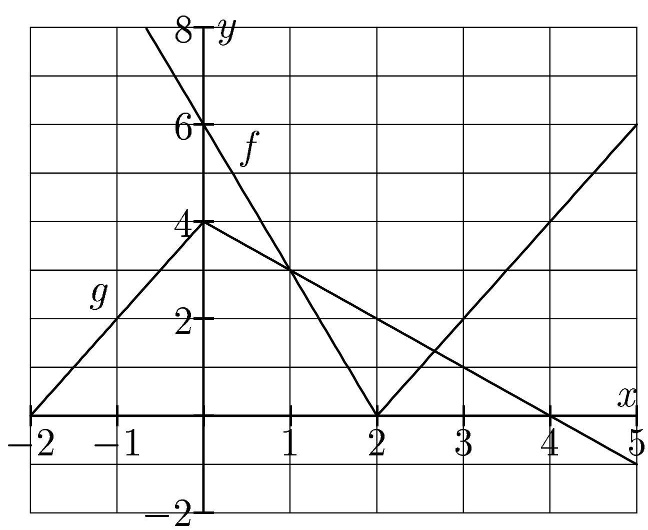
(b) Find $C'(200)$. Give an interpretation in words describing the meaning of your answer. (7 pts.)

Answer: $C'(200) = 5 - 16 + 36 = 25\$$ per TV. This means the cost of producing the 201st plasma TV is approximately \$25. Alternatively, it costs an additional \$25 to produce the 201st television set, so that $C(201) - C(200) \approx 25$. In fact, one can check that

$$C(201) - C(200) = 4,325.1403 - 4300 = 25.1403$$

so that our approximation of \$25 is quite good.

4. The graphs of the functions $f(x)$ and $g(x)$ are given below. Suppose that $h(x) = f(x)/g(x)$ and $C(x) = f(g(x))$. If they exist, find both $h'(3)$ and $C'(3)$. (14 pts.)



Answer:

To find $h'(3)$, use the quotient rule and then find the relevant information from the graphs of f and g. Note that the derivatives can be found by calculating the slope of the relevant line.

We have

$$
\begin{aligned}
h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \implies \\
h'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} \\
&= \frac{1 \cdot 2 - 2 \cdot (-1)}{(1)^2} \\
&= 4
\end{aligned}
$$

To find $C'(3)$, use the chain rule and then find the relevant information from the graphs of f and g. We have

$$
\begin{aligned}
C'(x) &= f'(g(x)) \cdot g'(x) \implies \\
C'(3) &= f'(g(3)) \cdot g'(3) \\
&= f'(1) \cdot g'(3) \\
&= -3 \cdot -1 \\
&= 3
\end{aligned}
$$

5. Suppose that $f(x) = 2x^2 - x^4$. Feel free to use the back of this page for work/answers if necessary. (24 pts.)

(a) Find the intervals on which f is increasing or decreasing.

Answer: We compute

$$f'(x) = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 - x)(1 + x)$$

so that f has critical points at $x = -1, 0, 1$. By drawing a number line and checking values on either sides of the critical points, we see that f' is positive when $x < -1$ and $0 < x < 1$ and f' is negative for $-1 < x < 0$ and $x > 1$. It follows that f is increasing when $x < -1$ and $0 < x < 1$ and decreasing when $-1 < x < 0$ and $x > 1$. The pattern of the signs on the first derivative number line alternates $+-+-$ because the factors in the derivative are all raised to an odd power.

(b) Find the intervals on which f is concave up or concave down.

Answer: Next, we have that

$$f''(x) = 4 - 12x^2 = 4(1 - 3x^2).$$

Solving $1 - 3x^2 = 0$ for x gives $x^2 = 1/3$ or $x = \pm 1/\sqrt{3}$. These are possible inflection points. Checking the signs of the second derivative on either side of these values shows that f'' is positive for $-1/\sqrt{3} < x < 1/\sqrt{3}$ (concave up) and f'' is negative for $x < -1/\sqrt{3}$ and for $x > 1/\sqrt{3}$ (concave down). The sign pattern on the second derivative number line is $- + -$.

(c) Find any local maxima, local minima or inflection points.

Answer:
Using part **(a)**, we know that $x = -1$ and $x = 1$ are each local max's because f is increasing directly to their left and decreasing directly to their right (first-derivative test). This also follows since f is concave down at these points by part **(b)**. The critical point $x = 0$ is a local min because the function decreases to the immediate left of 0 and increases to the immediate right. Computing the function values at these points, we have

$$\text{local max's at } (-1, 1), (1, 1), \qquad \text{local min at } (0, 0).$$

By part **(b)**, $x = \pm 1/\sqrt{3}$ are inflection points. The y-coordinates for these points can be found by noticing that

$$f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}.$$

Thus, the two inflection points are $(-1/\sqrt{3}, 5/9)$ and $(1/\sqrt{3}, 5/9)$.

(d) Use the information from parts **(a)**, **(b)** and **(c)** to sketch the graph of f.

Answer: Note that f is an even function, so our graph is symmetric about the y-axis.