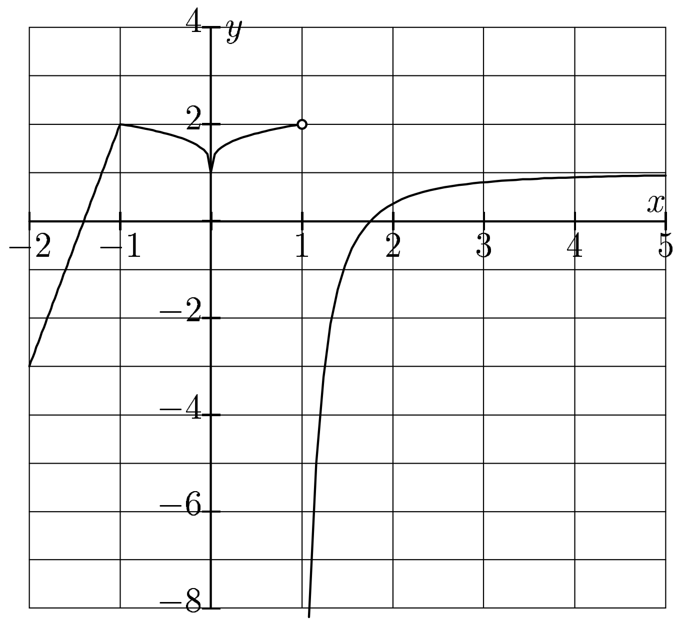
MATH 125 Calculus for the Social Sciences I

Exam #2 SOLUTIONS **Prof. G. Roberts**

1. The graph of $f(x)$ is shown below. Use it to answer each of the following questions. Note that ∞ or $-\infty$ are acceptable answers. (18 pts.)



(a) Evaluate $\lim\limits_{x \to -2^+} f(x)$ **Answer:** -3

(b) Evaluate $\lim\limits_{x \to 1^-} f(x)$ **Answer:** 2

(c) Evaluate $\lim\limits_{x \to 1^+} f(x)$ **Answer:** $-\infty$

(d) Evaluate $\lim\limits_{x \to \infty} f(x)$ **Answer:** 1

(e) List any numbers where f is **not** continuous.
Answer: $x = 1$ (function value does not exist)

(f) List any numbers where f is **not** differentiable.
Answer: $x = -1$ (corner), $x = 0$ (cusp), $x = 1$ (discontinuity)

2. Evaluate each of the following limits, if they exist. Note that ∞ or $-\infty$ are acceptable answers. Be sure to show your work. (6 pts. each)

(a) Evaluate $\lim_{x\to\pi}(\cos(x)+\sin(x))$

Answer:
$$\lim_{x\to\pi}(\cos(x)+\sin(x)) = \lim_{x\to\pi}\cos(x) + \lim_{x\to\pi}\sin(x) = \cos(\pi)+\sin(\pi)$$
$$= -1+0 = -1.$$

(b) Evaluate $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

Answer: Multiply top and bottom by the conjugate.

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)}$$
$$= \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{4}.$$

(c) Evaluate $\lim_{x\to\infty}\frac{2x^3-3x^2+4}{5x^3+7x-\pi}$

Answer: Divide top and bottom by x^3 to obtain

$$\lim_{x\to\infty}\frac{\frac{2x^3}{x^3}-\frac{3x^2}{x^3}+\frac{4}{x^3}}{\frac{5x^3}{x^3}+\frac{7x}{x^3}-\frac{\pi}{x^3}} = \lim_{x\to\infty}\frac{2-\frac{3}{x}+\frac{4}{x^3}}{5+\frac{7}{x^2}-\frac{\pi}{x^3}} = \frac{2}{5}$$

since

$$\lim_{x\to\infty}\frac{a}{x^n} = 0 \quad \text{for any } n>0.$$

3. **(a)** Give a limit definition of the derivative of a function $f(x)$. (5 pts.)

Answer:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use your limit definition of the derivative from part **(a)** to find the derivative of the function $f(x) = 3x^2 - 4x$. (10 pts.)

Answer:

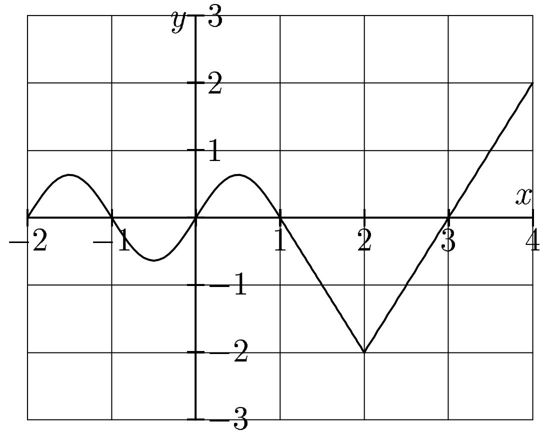
$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{3(x+h)^2 - 4(x+h) - (3x^2 - 4x)}{h} \\
&= \lim_{h \to 0} \frac{3(x^2 + 2xh + h^2) - 4x - 4h - 3x^2 + 4x}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 4x - 4h - 3x^2 + 4x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 4h}{h} \\
&= \lim_{h \to 0} \frac{h(6x + 3h - 4)}{h} \\
&= \lim_{h \to 0} 6x + 3h - 4 \\
&= 6x - 4
\end{aligned}$$

(c) Suppose that the displacement of a particle (in feet) is given by the function $s(t) = 3t^2 - 4t$ where the time t is measured in seconds. (This is the same function as part **(b)** with different variables.) What is the instantaneous velocity at time $t = 3$? (5 pts.)

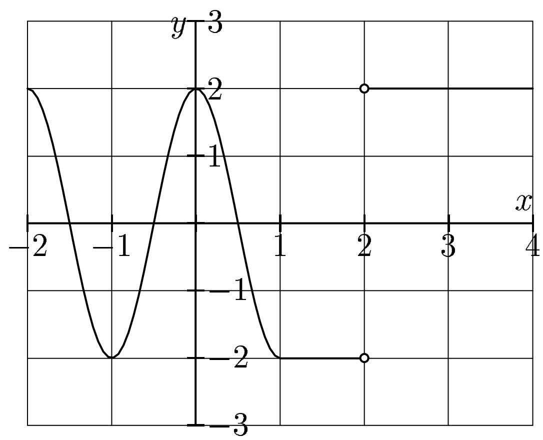
Answer:

Using the answer to part **(b)**, we have that $v(t) = s'(t) = 6t - 4$, since instantaneous velocity is given by the slope of the tangent line of the position function $s(t)$. Therefore, $v(3) = 6(3) - 4 = 14$ feet per second.

4. Given the graph of the function $g(x)$ below, sketch the graph of the derivative $g'(x)$ on the axes provided. (12 pts.)



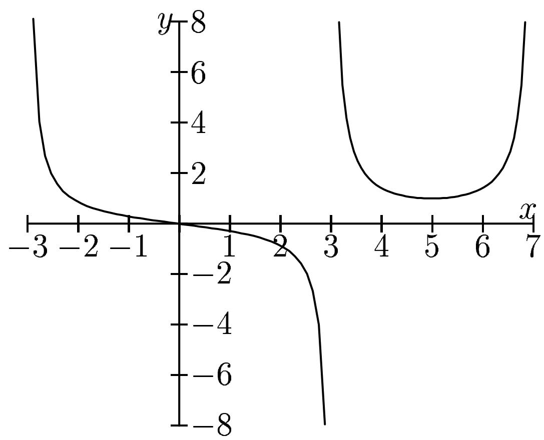
Answer:



5. Sketch the graph of a function $g(x)$ satisfying all of the following properties: (12 pts.)

- g is continuous at all x except for $x = 3$
- $g(0) = 0$ and $g(3)$ does not exist
- $g'(5) = 0$
- $\lim_{x \to 3^-} g(x) = -\infty$
- $\lim_{x \to 3^+} g(x) = \infty$
- $g'(x) < 0$ for $x < 3$ or $3 < x < 5$
- $g'(x) > 0$ for $x > 5$
- $g''(x) < 0$ for $0 < x < 3$
- $g''(x) > 0$ for $x < 0$ or $x > 3$

Answer:



6. Some final conceptual questions. You must **show your work** to receive any partial credit. (20 pts.)

(a) If $x^2 - 3x \leq f(x) \leq -7x - 4$ for all x, find $\lim_{x \to -2} f(x)$.

Answer: 10. The functions above and below $f(x)$ are both approaching 10 as x approaches -2. In other words,

$$\lim_{x \to -2} x^2 - 3x \;=\; \lim_{x \to -2} -7x - 4 \;= 10.$$

By the Squeeze Theorem, we conclude that $\lim_{x \to -2} f(x) = 10$ as well.

(b) Use the Intermediate Value Theorem to show that the equation $\ln x = x/4$ has a solution between 1 and e.

Answer: The key step here is to write the equation in the form $\ln x - x/4 = 0$ so that there is a constant on the right-hand side. Next, define $f(x) = \ln x - x/4$ and compute $f(1) = \ln 1 - 1/4 = -1/4$ and $f(e) = \ln e - e/4 = 1 - e/4 > 0$. Thus, we have that

$f(1) < 0$ while $f(e) > 0$. By the Intermediate Value Theorem, since f is continuous, there must be a number c between 1 and e such that $f(c) = 0$ or $\ln c - c/4 = 0$, as desired.

(c) Suppose that $P(n)$ represents the profit earned in dollars for selling n stereos. Which of the following best describes the meaning of $P'(500) = 100$?

(i) The profit earned from selling 100 stereos is \$500.
(ii) The profit earned from selling 500 stereos is \$100.
(iii) Selling the 501st stereo will earn, approximately, an additional \$100 in profit.
(iv) Selling the 101st stereo will earn, approximately, an additional \$500 in profit.
(v) The rate of change of the profit is \$500 per stereo after selling 100 stereos.

Answer: (iii) $P'(500) = 100$ means the rate of change in the function when $n = 500$ is 100 dollars per stereo. Thus, selling the next stereo (the 501st), will yield, roughly, another \$100 in profit.

(d) A stock analyst gives the following advice: "Although the value v of a share in Roberts Inc. continues to decline, it is declining at a slower rate. It might be wise **not** to completely sell off the stock." Interpret this statement in terms of the signs of $v'(t)$ and $v''(t)$, where $v(t)$ is the value of the stock at time t.

Answer: The graph of $v(t)$ will be decreasing since the stock price is going down. However, it will also be concave up since it is going down at a slower rate (the slopes are increasing in value, becoming less negative). Therefore, the first derivative $v'(t)$ is negative while the second derivative $v''(t)$ is positive.