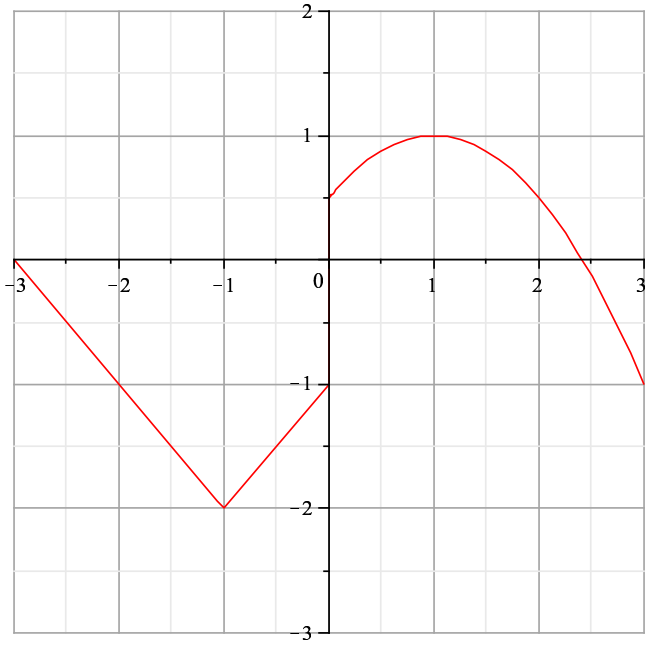
MATH 125-01/02 Calculus for the Social Sciences I

Exam #1 September 27, 2007 Prof. G. Roberts

SOLUTIONS

1. The **entire** graph of $f(x)$ is shown below. Use it to answer each of the following questions: (26 pts.)



(a) What is the domain of f? $[-3, 3]$

(b) What is the range of f? $[-2, 1]$

(c) Is f a one-to-one function? Explain.

No, f is not one-to-one because it fails the horizontal line test. There are elements in the range that have more than one pre-image in the domain. For example, -1 has three inputs in the domain that are sent to it.

(d) For what value(s) of x does $f(x) = -1$? Drawing a horizontal line $y = -1$, we find three points of intersection: $x = -2, 0$ and 3.

(e) What is $(f \circ f)(-1)$? -1

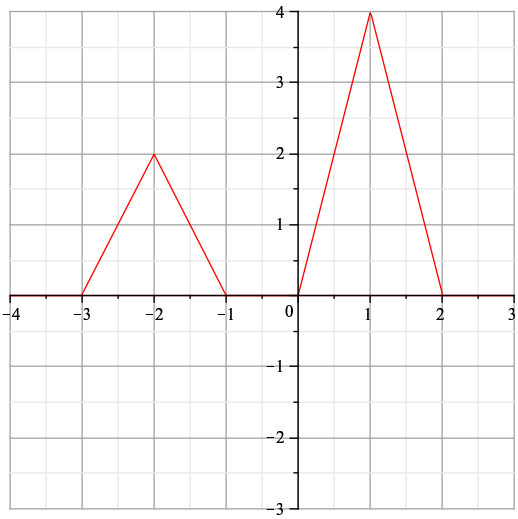
$(f \circ f)(-1) = f(f(-1)) = f(-2) = -1$.

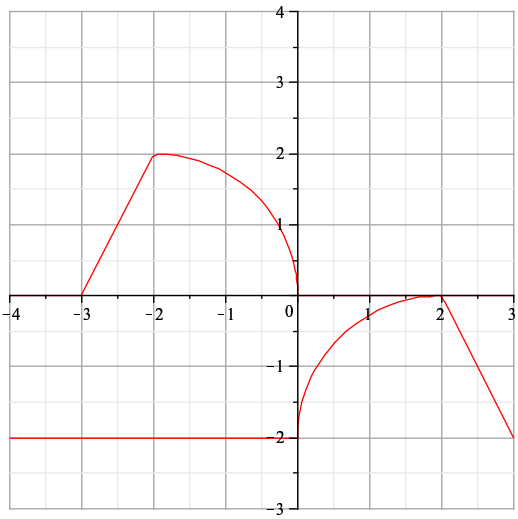
(f) The $\lim_{x \to 0^+} f(x) = 0.5$.

(g) The $\lim_{x \to 0^-} f(x) = -1$.

(h) The $\lim_{x \to 0} f(x)$ does not exist. The left-hand and right-hand limits are different so the limit does not exist.

2. For each of the graphs shown below, one or more transformations have been applied to the original function $f(x)$ (on the left) to obtain a new function $g(x)$ (on the right). In mathematical terms, state the formula for $g(x)$ in terms of $f(x)$. For example, a typical answer might be $g(x) = 3f(2x+1)$. (12 pts.)





(a) $g(x) =$ $2f(x-3)$ **(b)** $g(x) =$ $f(-x)-2$

For **(a)**, g is obtained by shifting f right by 3 units (replace x by $x-3$) and then stretching vertically by a factor of 2 (multiply on the outside by 2). For **(b)**, g is obtained by reflecting f about the y-axis (replace x by $-x$) and then shifting vertically down by two units (subtract 2 on the outside).

3. The table below shows the entries for a linear function, an exponential function and a trigonometric function.

x	f(x)	g(x)	h(x)
-1	1	12	12
0	0	6	6
1	-1	3	0
2	0	1.5	-6
3	1	0.75	-12

(a) Match each of the functions f, g or h with its correct type (linear, exponential or trigonometric). **Explain your answer thoroughly.** (6 pts.)

f is trigonometric because it's values oscillate between 1 and -1.

g is exponential because each successive value is obtained by **multiplying** the previous entry by a constant factor of $1/2$. Its graph resembles exponential decay.

h is linear because each successive value is obtained by subtracting 6 from the previous entry, giving a constant slope of -6. Its graph is a line.

(b) Find a formula for the linear function. (5 pts.)

Slope: $\frac{12-6}{-1-0} = -6$ and y-intercept: 6 (since $h(0) = 6$)

Therefore, $h(x) = -6x + 6$

(c) Find a formula for the exponential function. (5 pts.)

Use $g(x) = c \cdot a^x$ and find c and a. Two good points to use are $(0, 6)$ and $(1, 3)$. Using $g(0) = 6$, we find

$$6 = c \cdot a^0 = c$$

and using $g(1) = 3$, we find

$$3 = 6 \cdot a^1 \quad \text{which implies } a = 1/2.$$

Thus, $g(x) = 6 \cdot (\frac{1}{2})^x$.

(d) Fill in the bottom row of the table. (6 pts.)

$f(3) = 1$ (by inspection), $g(3) = 6 \cdot (\frac{1}{2})^3 = \frac{3}{4}$ (or divide the previous entry by 2) and $h(3) = -18 + 6 = -12$ (or subtract 6 from the previous entry).

4. A population of mold has begun to develop on a piece of bread you brought back to your dormitory from Kimball dining hall. Initially, the population is only 1000 cells, but after two hours, it has **quadrupled** to 4000 cells. Assume the cells continue to grow exponentially at this rate. Use the variables t for time and P for population of mold cells.

(a) How many mold cells are on the bread after six hours? $64,000$

One approach is to make a table continuing the exponential growth of quadrupling every two hours (multiply by 4).

t	P
0	1,000
2	4,000
4	16,000
6	64,000

(b) Find an expression in the form $P(t) = c \cdot a^t$ modeling the number of mold cells on the bread after t hours. $P(t) = 1000 \cdot 2^t$

Since the initial population is 1000, $c = P_0 = 1000$. Using the point $(2, 4000)$ gives

$$4000 = 1000 \cdot a^2 \quad \text{which implies } a^2 = 4 \text{ or } a = 2.$$

(c) Public Safety will confiscate your bread if the number of mold cells reaches one million. When will this happen? Give the **exact** answer as well as an approximation rounded to two decimal places. The exact answer is $\log_2 1000$ or $\dfrac{\ln 1000}{\ln 2}$. The approximate answer is 9.97 hours.

Using our formula from part **(b)**, we must solve

$$1000000 = 1000 \cdot 2^t$$

for t. Divide both sides by 1000 to obtain $1000 = 2^t$. This is precisely the definition of $\log_2 1000$. However, to use a calculator, take the natural logarithm of both sides to obtain

$$\ln 1000 = \ln 2^t = t \ln 2.$$

Divide both sides by $\ln 2$ to get $t = \dfrac{\ln 1000}{\ln 2} \approx 9.97$ hours.

5. Some final conceptual questions. You must **show your work** to receive any partial credit. (25 pts.)

(a) The function $g(x) = x^5 - 3x^2 + x$ is <u>neither odd nor even</u>.

(b) Simplify $\log_3 \frac{1}{9} + \ln e^7 = -2 + 7 = 5$.

To find $\log_3 \frac{1}{9}$, solve $3^x = \frac{1}{9}$ for x. Since $3^2 = 9$, we have $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$ and thus $\log_3 \frac{1}{9} = -2$. Using the fact that $\ln x$ and e^x are inverses, we have that $\ln e^x = x$ so that $\ln e^7 = 7$.

(c) Suppose that $C(t)$ is a function measuring the cost C of a bleacher seat at Fenway Park in the year t, where $t = 0$ corresponds to the year 1960. Which of the following best describes the meaning of $C^{-1}(6) = 30$?

(iv) In 1990, the price of a bleacher seat was \$6.

This was a question on understanding inverses. The function $C(t)$ takes as input time and outputs cost. Thus, the inverse (going the other way), takes as input cost and outputs time. Therefore, $t = 30$ and $C = 6$ so that in thirty years, the cost will be \$6.

(d) The $\lim\limits_{x \to 2} \dfrac{5x - 10}{x^2 - 4} = \dfrac{5}{4}$.

$$\lim_{x \to 2} \frac{5x - 10}{x^2 - 4} = \lim_{x \to 2} \frac{5(x - 2)}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{5}{x + 2} = \frac{5}{4}.$$

(e) If $s(t) = 3t^2 - t$ represents the distance in feet a ball has traveled after t seconds, then the average velocity over the interval $[2, 5]$ is <u>20 feet/sec</u>.

The average velocity is just the slope of the secant line between the two points $(2, s(2))$ and $(5, s(5))$. This is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{70 - 10}{3} = 20 \text{ feet/sec}.$$