Practice Problems 1 – 14

1. a. PMF of a dice

$p(X=1) = 1/6$ $\quad$ $P(X=4) = 1/6$

$p(X=2) = 1/6$ $\quad$ $P(X=5) = 1/6$

$p(X=3) = 1/6$ $\quad$ $p(X=6) = 1/6$

b. $E(X) = \sum X \ p(X)$

$= 1 \cdot 1/6 + 2 \cdot 1/6 + 3 \cdot 1/6 + 4 \cdot 1/6 + 5 \cdot 1/6 + 6 \cdot 1/6$

$\boxed{E(X) = 3.5}$

c. $E(X^2) = \sum x^2 \cdot p(X)$

$1 \cdot 1/6 + 4 \cdot 1/6 + 9 \cdot 1/6 + 16 \cdot 1/6 + 25 \cdot 1/6 + 36 \cdot 1/6$

$= 15.167$

d. $Var(X) = |E(X^2) - E(X)^2|$

$= 15.167 - 3.5^2$

$= \boxed{2.9167}$

2. Poisson Distribution

$$\lambda = 6 \qquad k = 5$$

$$P(\lambda) = \frac{\lambda^k \cdot e^{-\lambda}}{k!} = \frac{6^5 \cdot e^{-6}}{5!} = 0.161$$

3. ~~No~~ To ask exactly 48 people to get 42 successes, we need 41 successes in the first 47 people and a success in the 48th person

$$P(X=41) = \binom{47}{41} \cdot 0.9^{41} \cdot 0.1^{6}$$ ← binomial Distribution

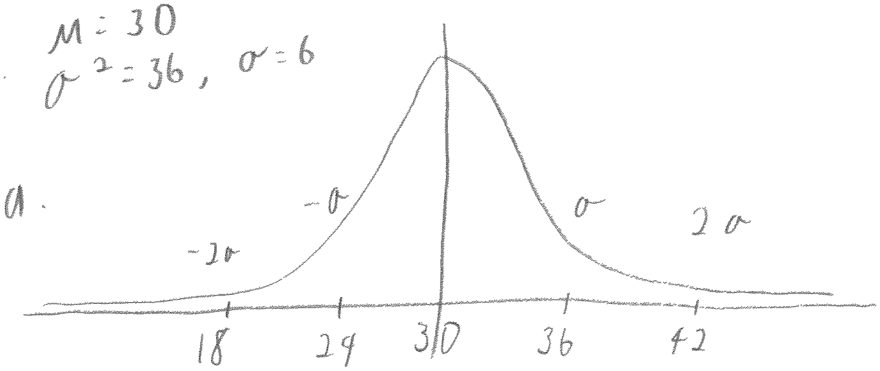
$$= 0.143$$

$$P(48^{th} \text{ trial is success}) = 0.9$$

$$P(X=41) \mid P(48^{th} \text{ trial success}) = 0.143 \cdot 0.9 = \boxed{0.129}$$

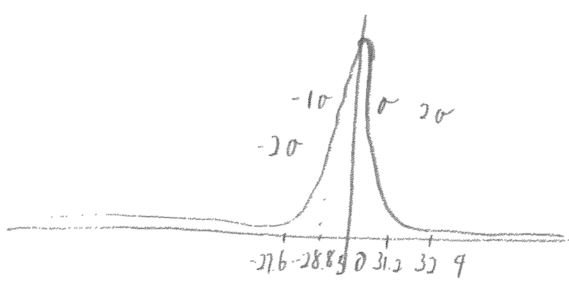
4. $\mu = 30$
$\sigma^2 = 36, \; \sigma = 6$

a.



b. By CLT, $\sigma^2_{mean} = \dfrac{\sigma^2_{sample}}{n} = \dfrac{36}{25} = 1.44$

$\sigma_{mean} = 1.2$



Much narrower distribution because the spread of the mean of a sample is less than the spread for a single sample

5. TBD

6. t- distribution

19 degrees of freedom

a. For $\mu \neq 12.4$, $t_{crit} = \pm 2.093$ $(t_{\alpha/2})$

$$t = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{11.30 - 12.40}{2.48/\sqrt{20}} = \frac{-1.1}{0.55} = -1.99$$

Since $|t| < |t_{crit}|$, there is insufficient evidence to reject H_0

b. For $\mu < 12.4$ $t_{\alpha} = -1.73$

~~suffici~~ Since $|t| > |t_{crit}|$, there is enough evidence to reject $\mu < 12.4$

7. $\mu = n \cdot p = 40 \cdot 0.55 = 22$

$$\sigma = \sqrt{n \cdot p \cdot (1-p)} = 9.9$$

8. $f(x) = \frac{d\,\Phi(x)}{dx}$, where $f(x)$ is the pdf and $\Phi(x)$ is the CDF

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

a. $\mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx$

$$= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \boxed{\frac{b+a}{2}}$$

b. $\frac{1}{2} = \int_{-\infty}^{median} f(x)\,dx$

$\frac{1}{2} = \int_a^{median} \frac{1}{b-a}\,dx$

$\frac{1}{2} = \frac{median - a}{b-a}$

$b - a = 2 \cdot median - 2a$

$\boxed{\frac{b+a}{2} = median}$

c. Variance $= \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \mu^2$

$$= \int_a^b \frac{x^2}{b-a}\, dx - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{x^3}{3(b-a)}\Big|_a^b - \frac{(b+a)^2}{4}$$

$$= \frac{b^3 - a^3}{3b - 3a} - \frac{(b+a)^2}{4} = \frac{1}{12}(b-a)^2$$

9. Assume a binomial distribution; 50% heads, 50% tails.

However, because there is 100 samples, using CLT we can assume normal distribution by CLT.

$\mu = 0.5 \cdot 100 = 50$

$\sigma^2 = p \cdot (1-p) \cdot n = 25$

$\sigma = 5$

40-60 heads is ± 2 standard deviations

$P(-2 \le z \le 2) = \boxed{0.95}$

~~13~~.

12. $\cancel{p(x) = p(\text{first flag } o}$

$$\cancel{p = 1 \cdot \frac{12}{13} \cdot \frac{11}{12} \cdot \frac{10}{11} \cdot \frac{9}{10}}$$

$$p = \underset{\substack{\text{first flag}\\\text{can be}\\\text{on any pole}}}{1} \cdot \underbrace{\frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \cdot \frac{8}{12}}_{\substack{\text{subsequent flags cannot}\\\text{be on same pole}\\\text{as previous flags}}} = 0.382$$

first flag can be on any pole

~~less pos~~ subsequent flags cannot be on same pole as previous flags

No differenie if ~~poles~~ flags ~~indinguishible~~ indistinguishible

14. $$p(t) = \frac{1}{1000} \cdot e^{-\frac{t}{1000}} \quad \text{if } t > 0$$

$$0 \quad \text{if } t \leq 0$$

$$P(t \geq 2000) = \int_{2000}^{\infty} p(t)\, dt$$

$$= \int_{2000}^{\infty} \frac{1}{1000} e^{-\frac{t}{1000}}\, dt$$

$$= -e^{-\frac{t}{1000}} \Big|_{2000}^{\infty}$$

$$= \cancel{0} - -.135$$

$$\boxed{P(t \geq 2000) = 0.135}$$

10. $P(3 \text{ hearts} \mid 3 \text{ red}) = \dfrac{\binom{13}{3}}{\binom{26}{3}}$ $\leftarrow$ choose 3 hearts from 13 / $\leftarrow$ choose 3 red from 26

$P(2 \text{ clubs} \mid 2 \text{ black}) = \dfrac{\binom{13}{2}}{\binom{26}{2}}$

$P(3 \text{ hearts} \cap 3 \text{clubs} \mid 3 \text{ red} \cap \text{black}) = \dfrac{\binom{13}{3}}{\binom{26}{3}} \cdot \dfrac{\binom{13}{2}}{\binom{26}{2}}$

11. $E(X) = \sum X \cdot p(X)$

$= \sum_{k=0}^{\infty} 2^k \cdot \dfrac{1}{2^{k+1}}$

$P(\text{first head on } k{+}1^{th} \text{ toss})$
$= P(\text{all tails up to } k \text{ tosses})$
$\cdot P(\text{head on } k{+}1 \text{ toss})$

$= \sum_{k=0}^{\infty} \dfrac{1}{2}$

$= \dfrac{1}{2^k} \cdot \dfrac{1}{2} = \dfrac{1}{2^{k+1}}$

$= \infty$

sorry wrong order)

13. $Z = \dfrac{X - \mu}{\sigma} = \dfrac{760 - 588}{94} = \boxed{1.83}$

$\text{Percentile}(Z = 1.83) = \boxed{.9664}$

(using tables)