Probability + Statistics
Final Review Answers - Part II

18. w/ 1000 samples, binomial dist. is basically normal

$$Z = \frac{y/n - p}{\sqrt{\frac{y}{n}(1-\frac{y}{n})/n}}$$

$$= \frac{5/1000 - 0.01}{\sqrt{\frac{5}{1000}(1-\frac{5}{1000})/1000}}$$

$$= -2.24$$

$$P(Z \leq -2.24)$$
$$= \Phi(-2.24)$$
$$= 1 - \Phi(2.24)$$
$$= 1 - 0.9875$$
$$= 0.0125$$

Redone @ end of Solutions

19. w/ 100 patients, dist is normal

p = likelihood a Patient is female = 0.667

$$\hat{p} = y/n$$
$$\hat{p} = 75/100$$
$$\hat{p} = 0.75$$

$$Z = \frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})/n}}$$

$$= \frac{0.75 - 0.667}{\sqrt{0.75(1-0.75)/100}}$$

$$= 1.9168$$

$$P(Z \geq 1.9168)$$
$$= 1 - \Phi(1.9168)$$
$$= 1 - 0.9726$$
$$= 0.0274$$

$$Z_{0.05} = 1.645$$

$$Z_{0.05} = \frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})/n}}$$

$$1.645 = \frac{0.7 - 0.667}{\sqrt{0.7(1-0.7)/n}}$$

$$n = 522$$

20. 8 days = 192 hours

$\theta = 96$

$$P(3 \text{ failures before } t = 192)$$
$$= P(1 \text{ failure before } t = 192)^3$$
$$= \left[\int_0^{192} f(x)\,dx\right]^3$$
$$= \left[\int_0^{192} \frac{1}{96} e^{-t/96}\,dt\right]^3$$
$$= 0.646$$

Redone @ End of Sol'ns

$\cancel{P(1 \text{ failure}) = \int_0^{192} \frac{1}{96} e^{-t/96} dt = 0.8647}$

21. $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

$= \frac{50-52}{3/\sqrt{10}}$

$= -2.1082$

$P(t(9) \leq -2.1082)$

$= 1 - P(t(9) \leq 2.1082)$

between 0.05 and 0.025

22. $\cancel{f(x) = \frac{\theta}{2} e^{-\theta|x|}}$

$\cancel{\ln f(x) = \ln(\theta/2) + -\theta|x|}$

$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2} e^{-\theta|x_i|}$

$= \left(\frac{\theta}{2}\right)^n e^{\sum_{i=1}^{n} -\theta|x_i|}$

$\ln(L(\theta)) = n \ln\left(\frac{\theta}{2}\right) + -\theta \sum_{i=1}^{n} |x_i|$

$= n \ln\theta - n\ln 2 - \theta \sum_{i=1}^{n} |x_i|$

$\frac{d}{d\theta}\left[\ln(L(\theta))\right] = \frac{n}{\theta} - \sum_{i=1}^{n} |x_i|$

$0 = \frac{n}{\theta} - \sum_{i=1}^{n} |x_i|$

$\sum_{i=1}^{n} |x_i| = \frac{n}{\theta}$

$\theta = n / \sum_{i=1}^{n} |x_i|$

$\frac{d^2}{d\theta^2} \ln(L) = -\frac{n}{\theta^2}$

always positive, always Positive

$= -n \cdot \frac{(\sum |x_i|)^2}{n^2}$

$\therefore$ 2nd derivative is always negative, so $\theta = \frac{n}{\sum_{i=1}^{n} |x_i|}$ is a maximum

23. a) assuming duration is normally distributed (when averaged over 30 patients) this is Kosher b/c of the central limit theorem

$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$= \frac{7-8}{4/\sqrt{30}}$

$= -1.3693$

$P(Z \leq -1.3693)$

$= 1 - \Phi(1.3693)$

$= 1 - 0.9147$

$= 0.0853$

b) $Z_2 = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$

$= \frac{9-8}{4/\sqrt{30}}$

$= 1.3693$

$P(-1.3693 \le Z \le 1.3693)$

$= \Phi(1.3693) - (1 - \Phi(1.3693))$

$= 2\Phi(1.3693) - 1$

$= \cancel{\ } \; 2 \cdot 0.9147 - 1$

$= 0.8294$

c) Assuming $\bar{X}$ is normally distributed b/c underlying distribution is normal

24. a) Assuming normal distribution and that the two distributions are independent

b) $t = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{n+m-2}\left(\frac{1}{n} + \frac{1}{m}\right)}}$

$= \frac{58-53}{\sqrt{\frac{(54-1)15^2 + (30-1)(13)^2}{54+30-2}\left(\frac{1}{54} + \frac{1}{30}\right)}}$

$= 1.53$

Using $r = n+m-2$
$= 82$

distr is basically normal

P(Z>1.53) = .06
Since this is a two-tailed test
p-value = P(Z>1.53)*2 = 0.12
c. Fail to reject

25. a) $t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$

$= \frac{47.1 - 48}{\sqrt{4.7/12}}$

$= -1.438$

Critical region $|t| \ge t_{\alpha/2}(r)$

$|t| \ge t_{0.05}(11)$

$|t| \ge 1.796$

Fail to reject

b) It's between $0.10 \le \frac{\alpha}{2} \le 0.05$

So p-value btw 0.2 and 0.1

c) $\bar{X} \pm t_{\alpha/2}(r) \cdot S_x/\sqrt{n}$

$47.1 \pm 1.796 \cdot \sqrt{4.7/12}$

$[45.976, 48.224]$

~~If we repeated the experiment many times, the true mean would be in our interval 90% of the time.~~

If we repeated the experiment many times, 90% of the CI's generated would contain μ (the true population mean)

26. critical region:

$$\frac{s_1^2}{s_2^2} \ge F_\alpha(n_1-1, n_2-1)$$

(Using $\alpha = 0.05$

$$\frac{1.04}{0.51} \ge 1.98$$ reject H_0

$$2.029 \ge 1.98$$

$P(\text{Rash}) = P(\text{M and Rash}) + P(\text{F \& R}$

$= 0.2 \cdot 0.9 + 0.8 \cdot 0.05$

27. $$P(\text{Flu} \mid \text{Rash}) = \frac{P(\text{Rash} \mid \text{Flu}) \cdot P(\text{Flu})}{P(\text{Rash})}$$

$$= \frac{(0.05)(0.8)}{(0.2)(0.9)+(0.8)(0.05)}$$

$$= 0.18$$

28. $P(\text{Tails}) = 0.5$

game ends on first tails

$p = 0.5$

Average duration $= \frac{1}{p} = 2$ flips

It's a geometric distribution

$$\text{Expected winnings} = \sum_{i=1}^{\infty} x_i f(x_i)$$

Expected value is infinite!

$$= \sum_{i=1}^{\infty} 2^{i-1} \left(\frac{1}{2}\right)^i$$

$$= \sum_{i=1}^{\infty} 2^{-1}$$

does not converge

29. $F(x) = \int_{-\infty}^{x} f(x)$

$= \int_{-\infty}^{x} \frac{1}{2}\alpha e^{-\alpha|x|} dx$

$= \frac{1}{2}\alpha \int_{-\infty}^{\infty} e^{-\alpha|x|} dx$

$= \frac{1}{2}\alpha \left[\int_{-\infty}^{0} e^{\alpha x} dx + \int_{0}^{\infty} e^{-\alpha x} dx\right]$

a) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$= \int_{-\infty}^{x} \frac{1}{2}\alpha e^{-\alpha|t|} dt$

$= \frac{1}{2}\alpha \int_{-\infty}^{x} e^{-\alpha|t|} dt$

$x \le 0$

$= \frac{1}{2}\alpha \int_{-\infty}^{x} e^{\alpha t} dt$

$= \frac{1}{2}\alpha \left[\frac{1}{\alpha} e^{\alpha t}\right]_{-\infty}^{x}$

$= \frac{1}{2}\alpha \left[\frac{1}{\alpha} e^{x\alpha}\right]$

$= \frac{1}{2} e^{\alpha x}$

$x \ge 0$

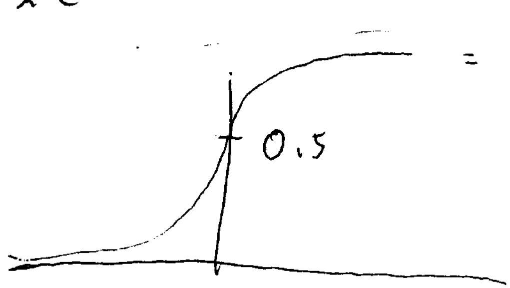
$\frac{1}{2}\alpha \left[\int_{-\infty}^{0} e^{\alpha t} dt + \int_{0}^{x} e^{-\alpha t} dt\right]$

$\frac{1}{2}\alpha \left[\frac{1}{\alpha} + \left[-\frac{1}{\alpha} e^{-\alpha t}\right]_{0}^{x}\right]$

$\frac{1}{2}\alpha \left[\frac{1}{\alpha} - \frac{1}{\alpha} e^{-\alpha x} + \frac{1}{\alpha}\right]$

$= \frac{1}{2}\alpha \left[\frac{2}{\alpha} - \frac{1}{\alpha} e^{-\alpha x}\right]$

$= 1 - \frac{1}{2} e^{-\alpha x}$



b) $\cancel{E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{2}\alpha e^{-\alpha|x|}\,dx}$

Since $F(0) = 0.5$, 0 is the mean of X

$$Var(X) = E[X^2] - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 \tfrac{1}{2}\alpha e^{-\alpha|x|}\,dx - 0$$

$$= \tfrac{1}{2}\int_{-\infty}^{0} x^2 \alpha e^{\alpha x}\,dx + \tfrac{1}{2}\int_{0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx$$

$$= \tfrac{1}{2}\int_{0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx + \tfrac{1}{2}\int_{0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx$$

integration by parts (see page 120 in textbook)

$$= \tfrac{1}{2}\cdot\left(\tfrac{2}{\alpha^2}\right) + \tfrac{1}{2}\cdot\left(\tfrac{2}{\alpha^2}\right)$$

$$= 2/\alpha^2$$

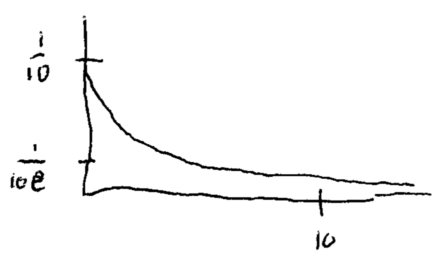
$$\therefore\ Std(X) = \sqrt{2}/\alpha$$

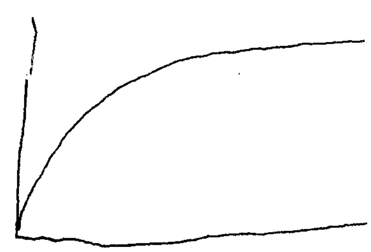
30. a) $f(x) = \frac{1}{\theta} e^{-x/\theta}$

$= \frac{1}{10} e^{-x/10}$

$F(x) = 1 - e^{-x/\theta}$

$= 1 - e^{-x/10}$





$P(X \geq 10)$

$= 1 - P(X < 10)$

$= 1 - F(10)$

$= 1 - [1 - e^{-10/10}]$

$= 1 - [1 - \frac{1}{e}]$

$= 1/e$

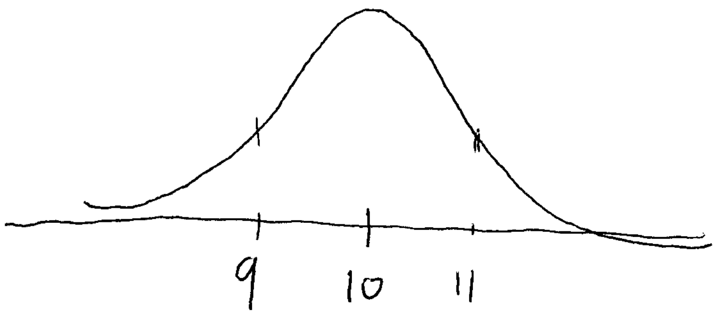
b) w/ 100 items, $\bar{X}$ will be basically normal by central limit thrm

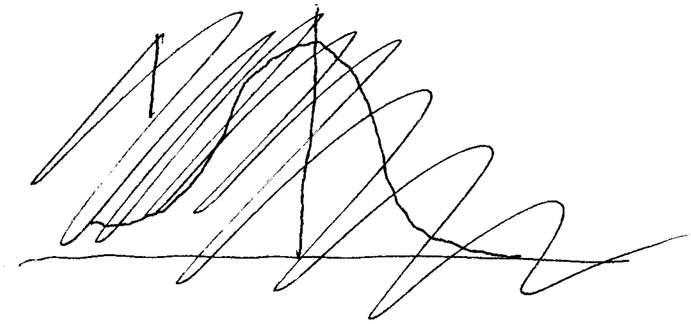
$\mu = 10$

~~Distribut~~

$\sigma_{\bar{x}} = \sigma_x / \sqrt{n} = 10/\sqrt{100} = 1$

$\bar{X} \sim N(\mu = 10, \sigma = 1)$





31. (See problem 23)

32. (See problem 25)

33. a) Assuming both distributions are normal. This is kosher b/c of the central limit theorem
Assuming both distributions are independent b/c they are composed of different individuals

b) $\hat{p}_1 = 337/500 = 0.674$

$\hat{p}_2 = 215/500 = 0.430$

$n_1 = 500$

$n_2 = 500$

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}}$$

$= 8.00$

P value ≈ 0

c) $\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\dfrac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \dfrac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

$z_{\alpha/2} = 1.96$

$[0.18, 0.30]$

d) Reject

34. a) Test statistic: $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$ $\quad df: n-1 = 10-1 = 9$

b) H_0: range $= 4\sigma$ $\quad H_a: \sigma > 10$

$40 = 4\sigma_0$

$\sigma_0 = 10$

c) $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

$= \dfrac{(9)(195)}{10^2}$

$= 17.55$

d) $\chi^2 \geq \chi^2_{\alpha}(n-1)$

$\geq \chi^2_{0.05}(9)$

≥ 16.92

e) $\left[\dfrac{(n-1)s^2}{\chi^2_{\alpha}(n-1)}, \infty\right)$

$\left[\dfrac{9 \cdot 195}{16.92}, \infty\right)$

$[103.72, \infty)$

f) REJECT

18. $P(\text{Errors} \leq 5)$

$= P(0 \text{ Error}) + P(1 \text{ Error}) + P(2 \text{ Error}) + P(3 \text{ Error}) + P(4 \text{ Error}) + P(5 \text{ Error})$

$$= \binom{1000}{0}(0.99)^{1000}(0.01)^0 + \binom{1000}{1}(0.99)^{999}(0.01)^1 + \binom{1000}{2}(0.99)^{998}(0.01)^2$$
$$+ \binom{1000}{3}(0.99)^{997}(0.01)^3 + \binom{1000}{4}(0.99)^{996}(0.01)^4 + \binom{1000}{5}(0.99)^{995}(0.01)^5$$

20. $P(\text{single computer fails before } 192 \text{ hours})$

$$= \int_0^{192} \frac{1}{96} e^{-t/96}\, dt$$

$= 0.8647$

$P(\text{at least 3 out of 5 computers fail})$

$\cancel{= P(0 \text{ fail}) + P(1 \text{ fail}) + P(2 \text{ fail}) + P(3 \text{ fail})}$

$\cancel{= \binom{5}{0}(1-0.8647)^5(0.8647)^0 + \binom{5}{1}(1-0.8647)^4(0.8647)^1}$

$\cancel{+ \binom{5}{2}(1-0.8647)^3(0.8647)^2 + \binom{5}{3}(1-0.8647)^2}$

$= P(5 \text{ fail}) + P(4 \text{ fail}) + P(3 \text{ fail})$

$$= \binom{5}{5}(0.8647)^5(1-0.8647)^0 + \binom{5}{4}(0.8647)^4(1-0.8647)^1$$
$$+ \binom{5}{3}(0.8647)^3(1-0.8647)^2$$