Probability + Statistics

Nov. 12 - In Class Problems

1. a) $L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} y_i e^{-y_i/\theta}$

$$= \frac{1}{\theta^{2n}} \left(\prod_{i=1}^{n} y_i\right) e^{-\frac{1}{\theta}\left(\sum_{i=1}^{n} y_i\right)}$$

$$\ell(\theta) = -2n\ln(\theta) + \ln\left(\prod_{i=1}^{n} y_i\right) - \frac{1}{\theta}\sum_{i=1}^{n} y_i$$

b) $\frac{d\ell}{d\theta} = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} y_i}{\theta^2} = 0 \implies \theta = \frac{\sum_{i=1}^{n} y_i}{2n} = \frac{n\bar{y}}{2n} = \frac{\bar{y}}{2}$

$\therefore \frac{\bar{y}}{2}$ is a critical point!

check concavity to verify it's a max

$$\frac{d^2\ell}{d\theta^2} = \frac{2n}{\theta^2} - \frac{2\sum_{i=1}^{n} y_i}{\theta^3}$$

$$= \frac{2n}{\theta^2} - \frac{2 \cdot 2n\theta}{\theta^3}$$

$$= \frac{2n}{\theta^2} - \frac{4n}{\theta^2}$$

Since n, θ are always positive, $\frac{d^2\ell}{d\theta^2}$ is always negative,

$\therefore \frac{\bar{y}}{2}$ is a maximum!

2. $L(\theta) = \prod_{i=1}^{n} (\theta+1) y_i^{\theta}$

$= (\theta+1)^n \left(\prod_{i=1}^{n} x_i\right)^{\theta}$

$l(\theta) = n \ln(\theta+1) + \theta \ln\left(\prod_{i=1}^{n} x_i\right)$

$\frac{dl}{\partial\theta} = \frac{n}{\theta+1} + \ln\left(\prod_{i=1}^{n} x_i\right) = 0$

$\theta + 1 = \frac{-n}{\ln(\prod x_i)}$

$\theta = -\left(\frac{n}{\ln(\prod x_i)} + 1\right)$

$\frac{d^2 l}{d\theta^2} = -\frac{n}{(\theta+1)^2}$

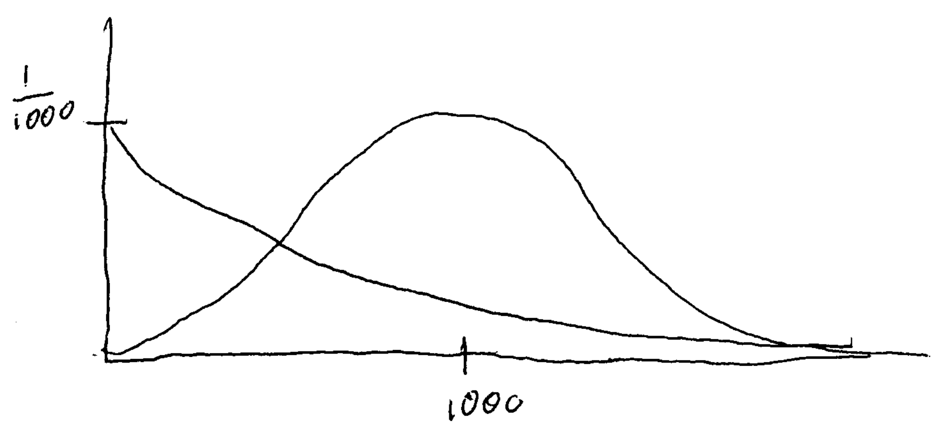
which is always negative. Thus, $\theta = -\left(\frac{n}{\ln(\prod x_i)} + 1\right)$ is a max

3. $f(T_i) = \begin{cases} 1/1000\, e^{-T_i/1000}, & T_i \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Mean for exponential is 1000

Variance for exponential is $(1000)^2$

$\bar{T} \sim N(\mu, \frac{\sigma^2}{\mu}) = N(1000, \frac{1,000,000}{n})$



4. $\sigma = 10$

$n = 100$

Find $P(|\bar{X} - \mu| < 1)$

$= P(-1 \le \bar{X} - \mu \le 1$

$= P\left(\frac{-1}{\sigma/\sqrt{n}} \le Z \le 1/\sigma/\sqrt{n}\right)$

$= P\left(\frac{-1}{1} \le Z \le \frac{1}{1}\right)$

$= 0.8413 - 0.1587$

$= 0.6826$

5. $\mu = 60 \quad n = 100$

$\sigma^2 = 64 \quad \sigma = 8$

$\bar{X} = 58$

Want to find $P(\bar{X} \le 58)$

(likelihood that a school just happened to be below 58 by chance)

$P\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \le \frac{58-60}{8/\sqrt{100}}\right)$

$= P(Z \le -2/.8)$

$= P(Z \le -2.5)$

$= 0.0062$ ←

Very low probability! Not a coincidence that HS has abnormally low scores