Post-Class Solutions Due 12 Feb

3.2

③ $P(|x|<1) = P(-1<x<1)$

$P(x^2 < 9/4) = P(-\frac{3}{2} < x < \frac{3}{2})$

a) $P(|x|<1) = \int_{-1}^{1} \frac{x^3+8}{32}\,dx = \boxed{\frac{1}{2}}$

$P(x^2<9/4) = \int_{-3/2}^{3/2} \frac{x^3+8}{32}\,dx = \boxed{\frac{3}{4}}$

b) $P(|x|<1) = \int_{-1}^{1} \frac{x+2}{18}\,dx = \boxed{\frac{2}{9}}$

$P(x^2<9/4) = \int_{-3/2}^{3/2} \frac{x+2}{18}\,dx = \boxed{\frac{1}{3}}$

⑦ Mean: $E[X]$

a) $E[X] = \int_0^1 x f(x)\,dx$
$= \int_0^1 6x^2(1-x)\,dx$
$= \boxed{\frac{1}{2}}$

$Var[X] = \int_0^1 (x-\mu)^2 f(x)\,dx$
$= \int_0^1 (x-1/2)^2\, 6x(1-x)\,dx$
$= \boxed{\frac{1}{20}}$

b) $E[X] = \int_1^{\infty} x f(x)\,dx$
$= \int_1^{\infty} \frac{2\cancel{x}}{x^{\cancel{3}2}}\,dx = \boxed{2}$

$Var[X] = \int_1^{\infty} (x-\mu)^2 f(x)\,dx$
$= \int_1^{\infty} (x-2)^2 \frac{2}{x^3}\,dx$

integral does not converge; variance does not exist.

c) $E[X] = \int_1^{\infty} x f(x)\,dx$
$= \int_1^{\infty} \frac{1}{x}\,dx$

integral does not converge; mean does not exist.

(11) $f(x) = F'(x)$

$$= \frac{d}{dx}\left(1+e^{-x}\right)^{-1}$$

$$= \frac{e^{-x}}{(e^{-x}+1)^2}$$

$$f(-x) = \frac{e^x}{(e^x+1)^2}$$

$$= \frac{e^x}{(e^x+1)^2} \cdot \frac{(e^{-x})^2}{(e^{-x})^2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} = f(x)$$

Graph is symmetric about $x=0$ because $f(x) = f(-x)$.

3.3

(1) Mean:

$$E[x] = \int_a^b x f(x)\,dx$$

$$= \int_a^b \frac{x}{b-a}\,dx$$

$$= \left.\frac{\frac{1}{2}x^2}{b-a}\right|_a^b$$

$$= \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2}{b-a} \quad \{\text{difference of squares}\}$$

$$= \frac{\frac{1}{2}(b+a)\cancel{(b-a)}}{\cancel{(b-a)}}$$

$$\boxed{\mu = \frac{1}{2}(b+a)}$$

$$\mu^2 = \frac{1}{4}(b+a)^2$$

$$Var[x] = E[x^2] - \mu^2$$

$$E[x^2] = \int_a^b x^2 f(x)\,dx$$

$$= \int_a^b \frac{x^2}{b-a}\,dx$$

$$= \left.\frac{\frac{1}{3}x^3}{b-a}\right|_a^b$$

$$= \frac{\frac{1}{3}b^3 - \frac{1}{3}a^3}{b-a} = \frac{(b^3-a^3)}{3(b-a)}$$

$$Var[x] = E[x^2] - \mu^2$$

$$= \frac{(b^3-a^3)}{3(b-a)} - \frac{(b+a)^2}{4}$$

factor difference of cubes + common denom.

$$= \frac{4\cancel{(b-a)}(b^2+ab+a^2)}{12\cancel{(b-a)}} - \frac{3(b+a)^2\cancel{(b-a)}}{12\cancel{(b-a)}} = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \boxed{\frac{(b-a)^2}{12} = \sigma^2}$$

⑦ Gamma

a) $\mu = \alpha\theta = 2\left(\frac{120}{7}\right) = \frac{240}{7} = \boxed{34.286}$ $\boxed{\bar{x} = 32.636}$

b) $\sigma^2 = \alpha\theta^2 = 2\left(\frac{120}{7}\right)^2 = \frac{28,800}{49} = \boxed{587.755}$

$\boxed{s^2 = 548.388}$

c) $P(x<35) = \boxed{0.605}$

$= \int_0^{35} \frac{1}{\Gamma(2)\left(\frac{120}{7}\right)^2} x e^{-x/\frac{120}{7}} \, dx$

$\Gamma(2) = (2-1)! = 1! = 1$

$\frac{13}{22}$ times are less than 35

$\boxed{\approx 0.591}$

⑨ $f(x) = \frac{1}{\theta^2} x e^{-x/\theta} \quad = \frac{1}{\theta^2}\left[x e^{-x/\theta}\left(\frac{-1}{\theta}\right) + e^{-x/\theta}\right] = \frac{e^{-x/\theta}}{\theta^2}\left(1 - \frac{1}{\theta}\right) = 0$

$\frac{x e^{-\frac{x}{\theta}}}{\theta^3} = \frac{e^{-x/\theta}}{\theta^2}$

$\theta^3 e^{-x/\theta} = \theta^2 e^{-x/\theta} x$

$\theta = x \; ; \; x = 2 \quad \theta = 2$

$P(x < 9.488) = 0.950$