A Short Introduction to Elementary Set Theory and Notation

Michael Moody, Fall 2007

Definition and Notation. Set theory is a relatively modern field and it's full development is at the heart of modern mathematics and logic. In this essay we will present a brief introduction to fundamental concepts, definitions, operations and set notation. The creator of set theory was the brilliant (and eccentric) German mathematician Georg Cantor (1845–1918). He introduced his theory of sets in a series of papers, the most famous of which were published in two parts (1895 and 1897) with the title *Beitäge zur Begründung der transfiniten Mengenlehre*. Cantor gave the following definition:

> By a "set" we mean any collection M into a whole of definite, distinct objects m (which are called the "elements" of M) of our perception (*Anschauung*) or of our thought.

Our current conception of a set has not fundamentally changed. Sets are well-defined collections of distinct "objects" that are called "members" or "elements" of the set; since Cantor mathematicians have conventionally denoted sets using capital letters[1]. By "well-defined" we mean that any given object can be unambiguously determined to either be a member of the given set, or not a member of the given set. Sets can be defined using language that establishes the conditions for membership in the set, or by explicitly listing the elements of the set, or often by giving a mathematical condition that must be met by elements of the set. Conventional notation uses braces to delimit the elements of a set. Several examples are listed below:

- A is the set of students at Olin College.
- B is the set of declared mechanical engineering majors at Olin College.
- C is the set of professors at Olin College.
- D is the set of integers whose squares are less than 10.
- $D = \{-3, -2, -1, 0, 1, 2, 3\}$.
- $D = \{n \mid n^2 < 10,\ n \text{ an integer}\}$.
- $E = \{n \mid n = k^3 \text{ for some integer } k\}$.

The last three examples show different ways to define the same set. The first definition of the set D is given using unambiguous language. The second definition explicitly defines D by listing the elements of the set. We point out that the order in which these elements are written does not matter—sets generally are considered to be unordered collections of elements. The last definition of D introduces the *set builder notation*. The vertical bar means "such that" and this description for D can be interpreted as "the set of all numbers n such that n is an integer and $n^2 < 10$." We use the symbol $\in$ to denote membership in a set, and the symbol $\notin$ to denote that an element is not a member of a given set; thus we that $2 \in D$ and $5 \notin D$, Dr. M $\in C$ and Dr. Adams $\notin A$.

[1] In the quoted definition Cantor denoted the set using the letter M, and used the lower case m to denote individual elements of the set; it is of mild interest to note that the German word for set is *Menge*, which begins with the letter M.

Set Relations, Set Operations and Some Special Sets. We say that set B is a *subset* of set A if every element of B is also an element of A. The symbol $\subset$ is used to indicate this relationship between two sets. Referring to our examples, since every declared mechanical engineering major at Olin is also a student at Olin, we write $B \subset A$ which is read as "B is a subset of A."

We say that two sets E and F are *equal* if both $E \subset F$ and $F \subset E$, in which case we write $E = F$. Thus, two sets are equal if they are subsets of each other—that is, they have exactly the same elements. If $E \subset F$ and $E \neq F$, then E is said to be a *proper subset* of F. Thus, if E is a proper subset of F then every element of E is contained in F and there is at least one element of F that is not contained in E. In our examples, B is a proper subset of A: not every student at Olin is a mechanical engineering major.

We define the *intersection* of set A and set B as the set of elements that are members of both A and B. We denote intersection using the symbol $\cap$ and we write $A \cap B$ for the intersection of A and B. Using set-builder notation we express the intersection this way: $A \cap B = \{m \mid m \in A \text{ and } m \in B\}$. Looking back at our examples, we see that $D \cap E = \{0, -1, 1\}$.

The concept of intersection leads us to define a special set. Returning again to our examples, we notice that there are no elements common to both set A and set C. We say that the intersection of these sets is *empty*—it is a set with no elements. This notion may seem to be paradoxical, but the need to ascribe emptiness to a set arises naturally, as suggested by our example: there are no faculty members that are students at Olin and no students at Olin are faculty members! We define the *empty set* as the set with no elements, and it is denoted either as $\{\,\}$ or, more commonly, as $\emptyset$. We can represent our example as $A \cap C = \emptyset$ or as $A \cap C = \{\,\}$.

We now define the *union* of sets A and B as the set of elements that are either in A or in B, or in both A and B; the use of the word *or* is therefore *inclusive.* The symbol for the union of sets is $\cup$, and we denote the union of sets A and B by $A \cup B$. Using set-builder notation we can write the definition of union: $A \cup B = \{m \mid m \in A \text{ or } m \in B\}$. Referring back to our examples, $A \cup B$ is the set of all people at Olin that are either a student or a faculty member.

The *cartesian product* of two sets A and B is a new set, denoted $A \times B$ and it is defined as follows: $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$. The cartesian product is thus the set of all ordered pairs of elements, the first of which is from set A and the second from set B.

The last concept that we will introduce is that of the *complement* of a set A relative to another set. Suppose that a given set A is a subset of another set Ω, which we write as $A \subset \Omega$. We define the *complement of A in Ω* as $\{m \in \Omega \mid m \notin A\}$. In words, it is the set of all elements of Ω that are *not* elements of A. Often the set Ω is implied, and we will write A^c for the compliment of A without explicitly referencing the set Ω.

Try the following exercise: Prove *De Morgan's Laws* for sets:

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^c = A^c \cup B^c$