Probability + Statistics
Post-class problems
November 9

$\boxed{3.2}$

#1 a) $P(A_1) = \int_0^1 \frac{x^3}{4}\,dx$

$= \left[\frac{1}{16}x^4\right]_0^1$

$= 1/16$

b) $P(A_2) = \int_{1/2}^{3/2} \frac{x^3}{4}\,dx$

$= \left[\frac{1}{16}x^4\right]_{1/2}^{3/2}$

$= 5/16$

c) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$= 1/16 + 5/16 - \int_{1/2}^{1} \frac{x^3}{4}\,dx$

$= 6/16 - \left[\frac{1}{16}x^4\right]_{1/2}^{1}$

$= 81/256$

#3 $P(|X|<1) = P(-1<X<1)$

$P(X^2 < 9/4) = P(-\frac{3}{2} < X < 3/2)$

a) $P(|X|<1) = \int_{-1}^{1} \frac{x^3+8}{32}\,dx = 1/2$

$P(X^2 < 9/4) = \int_{-3/2}^{3/2} \frac{x^3+8}{32}\,dx = 3/4$

b) $P(|X|<1) = \int_{-1}^{1} \frac{x+2}{18}\,dx = 2/9$

$P(X^2 < 9/4) = \int_{-3/2}^{3/2} \frac{x+2}{18}\,dx = 1/3$

#5 a) $\frac{1}{4} = \int_0^{q_1} 4x^3\,dx$ | $\frac{1}{2} = \int_0^{q_2} 4x^3\,dx$ | $\frac{3}{4} = \int_0^{q_3} 4x^3\,dx$

$\frac{1}{4} = x^4 \Big|_0^{q_1}$ | $\frac{1}{2} = x^4 \Big|_0^{q_2}$ | $\frac{3}{4} = x^4 \Big|_0^{q_3}$

$\frac{1}{4} = q_1^4$ | $\frac{1}{2} = q_2^4$ | $\frac{3}{4} = q_3^4$

$\frac{1}{\sqrt{2}} = q_1$ | $\frac{1}{\sqrt[4]{2}} = q_2$ | $\frac{3^{1/4}}{\sqrt{2}} = q_3$

b) $\frac{1}{4} = \int_{-\infty}^{q_1} \frac{1}{\pi(1+x^2)}\,dx$

$\frac{1}{4} = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{q_1}$

$\frac{1}{4} = \frac{1}{\pi}\tan^{-1}(q_1) + \frac{1}{2}$

$\frac{3}{4}\pi = \tan^{-1}(q_1)$

$-1 = q$

$\frac{1}{2} = \int_{-\infty}^{q_2} \frac{1}{\pi(1+x^2)}\,dx$

$\frac{1}{2} = \frac{1}{\pi}\tan^{-1}(x)\Big|_{-\infty}^{q_2}$

$\frac{1}{2} = \frac{1}{\pi}\tan^{-1}(q_2) + \frac{1}{2}$

$\pi = \tan^{-1}(q_2)$

$0 = q_2$

$\frac{3}{4} = \int_{-\infty}^{q_3} \frac{1}{\pi(1+x^2)}\,dx$

(Same logic)

$1 = q_3$

#7 a) $\cancel{E[X] = \int_0^1 x f(x)\,dx = \int_0^1 \frac{6x^2}{1-x}\,dx}$ ~~integral~~

$$E[X] = \int_0^1 x f(x)\,dx = \int_0^1 6x^2(1-x)\,dx = 1/2$$

$$Var[X] = \int_0^1 (x-\mu)^2 f(x)\,dx = \int_0^1 (x-1/2)^2\, 6x(1-x)\,dx = 1/20$$

b) $$E[X] = \int_1^\infty x f(x)\,dx = \int_1^\infty \frac{2}{x^2}\,dx = 2$$

$$Var[X] = \int_1^\infty (x-\mu)^2 f(x)\,dx = \int_1^\infty (x-2)^2 \frac{2}{x^3}\,dx$$

integral does not converge

c) $$E[X] = \int_1^\infty x f(x)\,dx = \int_1^\infty 1/x\,dx$$

integral does not converge

#11 $$f(x) = F'(x) = \frac{d}{dx}(1+e^{-x})^{-1} = \frac{e^{-x}}{(e^{-x}+1)^2}$$

$$f(-x) = \frac{e^{x}}{(e^{x}+1)^2} = \frac{e^{x}}{(e^{x}+1)^2}\cdot\frac{(e^{-x})^2}{(e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2} = f(x) \quad //$$

3.3

#1 $E[X] = \int_a^b x f(x)\,dx$

$$= \int_a^b \frac{x}{b-a}\,dx$$

$$= \frac{\frac{1}{2}x^2}{b-a}\Big|_a^b$$

$$= \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2}{b-a}$$

$$= \frac{\frac{1}{2}(b+a)(b-a)}{b-a}$$

$$= \frac{1}{2}(b+a)$$

$Var[X] = E[X^2] - \mu^2$

$$\cancel{= E\left[\frac{1}{b^2+2ab+a^2}\right] + \mu^2}$$

$$\cancel{= \int_a^b \frac{x}{(b-a)^2}\,dx - \frac{1}{4}(b+a)^2}$$

$$\cancel{= \frac{\frac{1}{2}x^2}{(b-a)^2}\Big|_a^b - \frac{1}{4}(b+a)^2}$$

$$\cancel{= \frac{\frac{1}{2}b^2 - \frac{1}{2}a^2}{(b-a)^2} - \frac{1}{4}(b+a)^2}$$

$$\cancel{= \frac{\frac{1}{2}(b-a)(b+a)}{(b-a)^2} - \frac{1}{4}(b+a)^2}$$

$$\cancel{= \frac{(b+a)}{2(b-a)} - \frac{1}{4}\frac{(b-a)(b+a)^2}{(b-a)}}$$

$$\cancel{= \frac{2(b+a) - (b-a)(b+a)^2}{4(b-a)}}$$

$$\cancel{= \frac{(b+a)(2-(b-a)(b+a))}{4(b-a)}}$$

$$\cancel{= \frac{(b+a)(2-b^2+a^2)}{4(b-a)}}$$

~~#3 $\int_{-\infty}^{w} a+(b-a)U(0,1)\,dx$~~

~~when $w<0$~~

~~$\int_{-\infty}^{w} a\,dx$~~

#3 a) $P[a+(b-a)Y \le w]$

$$= \frac{w-a}{b-a} \quad a \le w \le b$$

b) which is $U(a,b)$!

#7 a) $\mu = \frac{240}{7} = 34.286$, $\bar{X} = 32.636$

b) $\sigma^2 = 28{,}800/49$, $s^2 = 548.338$

c) $0.605 \approx 0.591 = 13/22$

#9 $\theta = 2$, 0.950

3.4

#3 a) 0.3849 b) 0.5403 c) 0.0603 d) 0.0013

e) 0.6826 f) 0.9544 g) 0.9974 h) 0.99

#5. $E[Y] = E[aX+b]$

$= aE[X]+b$

$= a\mu + b$

$Var[Y] = E[(Y-E[Y])^2]$

$= E[aX+b-(aE[X]+b))^2]$

$= E[(aX-aE[X])^2$

$= a^2 E[(X-E[X])^2]$

$= a^2 Var[X]$

$= a^2\sigma^2$

#9 $P[15.364 \le (X-7)^2 \le 20.096]$

$= P[\sqrt{15.364} \le X-7 \le \sqrt{20.096}] + P[\sqrt{15.364} \le 7-X \le \sqrt{20.096}]$

$= P[\sqrt{15.364}+7 \le X \le \sqrt{20.096}+7] + P[7-\sqrt{15.364} \ge X \ge 7-\sqrt{20.096}]$

$= 0.025$

#11. 0.514