Pre-Class Solutions for 12 Feb

3.4

(2) Use Table V

a) $z_{0.10} = 2.326$ b) $-z_{0.05} = -2.576$ c) $z_{0.0485} = 1.66$

d) $z_{0.9656} = 1.82$

(4) X is $N(650, 625)$ $\mu = 650,\ \sigma^2 = 625,\ \sigma = 25$

a) $P\left(\frac{600-650}{25} \le \frac{X-650}{25} \le \frac{660-650}{25}\right)$

$= \Phi(0.4) - \Phi(-2) = \boxed{0.6326}$

b) $P\left(-\frac{c}{25} < \frac{X-650}{25} < \frac{c}{25}\right) = 0.9544$

$\therefore \frac{c}{25} = 2,\ c = 50$

$\Phi\left(\frac{c}{25}\right) - 1 + \Phi\left(\frac{c}{25}\right) = 0.9544$

$2\Phi\left(\frac{c}{25}\right) = 1.9544$

$\Phi\left(\frac{c}{25}\right) = 0.9772$

$\boxed{c/25 = 2;\ c = 50}$

(10) Must Solve $f''(x) = 0$

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

$\ln(f(x)) = -\ln(\sigma\sqrt{2\pi}) - (x-\mu)^2/2\sigma^2$

$\frac{d}{dx}$: $\frac{f'(x)}{f(x)} = \frac{-2(x-\mu)}{2\sigma^2}$

$\frac{d}{dx}$: $\frac{f(x)f''(x) - [f'(x)]^2}{[f(x)]^2} = -\frac{1}{\sigma^2}$

$f''(x) = f(x)\left[\frac{-1}{\sigma^2} + \left(\frac{f'(x)}{f(x)}\right)^2\right] = 0$

$\frac{(x-\mu)^2}{\sigma^4} = \frac{1}{\sigma^2}$

$x - \mu = \pm\sigma$

$\boxed{x = \mu \pm \sigma}$

3.5

② Likelihood of a Gaussian, where $\theta = \sigma^2$ is product of Gaussian for each trial.

$$L(\theta) = \sqrt{\frac{1}{2\pi\theta}}\left(e^{-\frac{(x_1-\mu)^2}{2\theta}}\right) \cdot \sqrt{\frac{1}{2\pi\theta}}\left(e^{-\frac{(x_2-\mu)^2}{2\theta}}\right) \cdots \sqrt{\frac{1}{2\pi\theta}}\left(e^{-\frac{(x_n-\mu)^2}{2\theta}}\right)$$

$$L(\theta) = \left[\frac{1}{2\pi\theta}\right]^{\frac{n}{2}} \cdot e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\theta}} \quad 0 < \theta < \infty$$

$$\ln(L(\theta)) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\mu)^2$$

Take derivative, set to 0.

$$0 = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\boxed{\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \text{ for maximum likelihood}}$$

To show it's unbiased, we want to show $E(\hat{\theta}) = \theta = \sigma^2$

$$E(\hat{\theta}) = E\left(\frac{\theta}{n} \cdot \sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\theta}\right) = \frac{\theta}{n} \cdot n = \theta$$

added θ to numerator + denominator

average of these terms equals 1 by definition of variance.

⑥ Recall that $F(x)$ is the cumulative distribution function. The pdf $f(x) = F'(x) = \theta \cdot x^{-\theta-1}$ $\quad 1 \leq x \leq \infty$

$$L(\theta) = \theta \cdot x_1^{(-\theta-1)} \cdot \theta \cdot x_2^{(-\theta-1)} \cdot \theta \cdot x_3^{(-\theta-1)} \dots$$

$$= \theta^n (x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n)^{(-\theta-1)}$$

$$\ln(L(\theta)) = n \ln\theta + \ln(x_1 \cdot x_2 \cdots x_n) \cdot (-\theta-1)$$

$$\frac{d\ln(L(\theta))}{d\theta} = \frac{n}{\theta} - \ln(x_1 \cdot x_2 \cdots x_n) = 0$$

↑ we are taking the θ derivative.

$$\boxed{\theta = \frac{n}{\ln(x_1 x_2 \cdots x_n)}}$$

b/c 2nd derivative is $L(\theta) < 0$