Pre-class Problems

12-Nov-09

3.5

2. Likelyhood of a gaussian, where $\theta = \sigma^2$, is product of gaussian for each trial

$$L(\theta) = \sqrt{\frac{1}{2\pi\theta}}\, e^{-\frac{(x_1-\mu)^2}{2\theta}} \cdot \sqrt{\frac{1}{2\pi\theta}}\, e^{-\frac{(x_2-\mu)^2}{2\theta}} \cdots\cdots \sqrt{\frac{1}{2\pi\theta}}\, e^{-\frac{(x_n-\mu)^2}{2\theta}}$$

$$L(\theta) = \left[\frac{1}{2\pi\theta}\right]^{\frac{n}{2}} \cdot e^{-\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2\theta}} \qquad 0 < \theta < \infty$$

$$\ln L(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\mu)^2$$

Take the first derivative; set to 0

$$0 = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \quad \text{for max likelyhood}$$

To show it's unbiased, we want to show $E(\hat{\theta}) = \theta = \sigma^2$

$$E(\hat{\theta}) = E\left(\frac{\theta}{n}\cdot\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\theta}\right) = \frac{\theta}{n}\cdot n = \theta$$

added θ to num and den

average of these terms equals 1 by defn. of variance

6. Recall that F(x) is the cumulative distribution function

The pdf $f(x) = F'(x) = \theta \cdot x^{-\theta-1}$ $\quad 1 \le x < \infty$

$$L(\theta) = \theta \cdot x_1^{(-\theta-1)} \cdot \theta \cdot x_2^{(-\theta-1)} \cdot \theta \cdot x_3^{-\theta-1} \cdots$$

$$= \theta^n \cdot (x_1 \cdot x_2 \cdots x_n)^{-\theta-1}$$

$$\ln L(\theta) = n \ln \theta + \ln (x_1 \cdot x_2 \cdots x_n) \cdot (-\theta-1)$$

$$\frac{d}{d\theta}\left(\ln L(\theta)\right) = \frac{n}{\theta} - \ln (x_1 \cdot x_2 \cdots x_n) = 0$$

remember we are taking the θ-derivative

set as 0

$$\theta = \frac{n}{\ln(x_1 x_2 \cdots x_n)}$$

because 2nd derivative of $L(\theta) < 0$

3.6 4. $\mu = \int_0^2 x \cdot f(x)\, dx$

$$= \int_0^2 x \cdot \left(1 - \frac{x}{2}\right) dx$$

$$= \frac{x^2}{2} - \frac{x^3}{6} \Big|_0^2 = \boxed{\frac{2}{3}}$$

$$\sigma^2 = \int_0^2 x^2 f(x)\, dx - \mu^2$$

$$= \int_0^2 x^2 \left(1 - \frac{x}{2}\right) dx - \mu^2$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^2 - \frac{4}{9} = \boxed{\frac{2}{9}}$$

b. $P\left(\frac{2}{3} \le \bar{X} \le \frac{5}{6}\right) = P\left(\frac{\frac{2}{3}-\frac{2}{3}}{\sqrt{\frac{2}{9}/18}} \le \frac{\bar{X}-\frac{2}{3}}{\sqrt{\frac{2}{9}/18}} \le \frac{\frac{5}{6}-\frac{2}{3}}{\sqrt{\frac{2}{9}/18}}\right)$

σ of the ~~the~~ entire sample

$= P(0 \le Z \le 1.5) = \boxed{0.4332}$

6. a. $E(\bar{X}) = E(X) = 24.43$

b. $Var(\bar{X}) = \frac{Var(X)}{n} = \frac{2.2}{30} = 0.0733$

c $P\left(\frac{24.17 - E(\bar{x})}{\sigma(\bar{x})} \le Z \le \frac{24.82 - E(\bar{x})}{\sigma(\bar{x})}\right)$

$= P(-0.96 \le Z \le 1.44) = \boxed{0.7566}$

10. $E(\bar{x}) = E(X) = 2000$

$\sigma(\bar{x}) = \frac{\sigma(X)}{\sqrt{n}} = 100$

$P(\bar{X} > 2050)$

$= P\left(Z > \frac{2050 - 2000}{100}\right)$

$= \boxed{0.3085}$