MATH 363 – Sample Second Exam Question Solutions

April 25, 2011

The following are questions of the type that will appear on the second hour exam. The exam will have roughly the same number of parts, but may be organized differently.

1. True or false, with explanation or counterexample.

   (a) If a topological space X is path connected, then $\pi_1(X, p) = \{e\}$.

   False. The torus is path connected but has non-trivial fundamental group.

   (b) If a topological space X is contractible, then $\pi_1(X, p) = \{e\}$.

   True. If a space is contractible, it is homotopy equivalent to a point, so has the same fundamental group as a point, which is trivia.

   (c) If X is contractible and a topological group G acts on X, then $\pi_1(X/G, p) = \{e\}$.

   False. $\mathbb{R}^2$ is contractible and is acted on by $\mathbb{Z}^2$ to produce a torus as the quotient space. The fundamental group of the torus is $\mathbb{Z}^2$, which is non-trivial.

2. Let $X \subset \mathbb{R}^2$ be defined by

$$X = \{(x, y) : \ (x + 1)^2 + y^2 = 1\} \cup \{(x, y) : \ (x - 1)^2 + y^2 = 1\}.$$

   (See Figure 1.) Assume that X has the induced topology from $\mathbb{R}^2$.

   (a) Find a set of generators for the fundamental group of X.

   $\alpha(t) = (\cos(2\pi t) - 1, \sin(2\pi t))$, $0 \leq t \leq 1$ and $\beta(t) = (\cos(2\pi t - \pi) + 1, \sin(2\pi t - \pi))$, $0 \leq t \leq 1$ are generators tracing, respectively, the left circle and the right circle with base point the origin.

   (b) Using your generators, what form does a typical element of the fundamental group take?

   A typical element is a product of alternating powers (positive or negative) of $[\alpha]$ and $[\beta]$. For example, $[\alpha]^3[\beta]^{-2}[\alpha]^1[\beta]^1[\alpha]^{-2}$.

   (c) Is the fundamental group of X abelian? Explain.

   No, $[\alpha]$ and $[\beta]$ do not commute. That is, there is no homotopy between $\alpha \cdot \beta$ and $\beta \cdot \alpha$.

   (d) Let $f : X \to X$ be defined by $f(x, y) = (|x|, y)$. What is the induced homomorphism f_* on the fundamental group of X?

   f reflects the left circle onto the right, taking α to β^{-1} and leaving β fixed. In every expression of the form of part (b), powers of $[\alpha]$ are replaced by their reciprocals with α replaced by β. Thus, using the above as an example, $f_*([\alpha]^3[\beta]^{-2}[\alpha]^1[\beta]^1[\alpha]^{-2}) = [\beta]^{-3}[\beta]^{-2}[\beta]^{-1}[\beta]^1[\beta]^2 = [\beta]^3$

(e) Let $f : X \to X$ be defined by $f(x, y) = (x, |y|)$. What is the induce homomorphism f_* on the fundamental group of X?

The image of f is the lower half of the figure eight, which is contractible. Hence f_* is the trivial homomorphism mapping all elements to $[e]$.

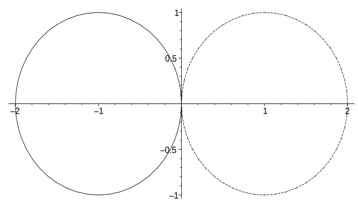

Figure 1:

3. Let $\pi : X \to Y$ be an identification map of path connected topological spaces. Let p be a base point in X and let $q = \pi(p)$ be a base point in Y.

   (a) What would it mean for π to have the path lifting property for paths that start at q?

   It would mean for each path α starting at q, $\alpha : [0, 1] \to Y$ with $\alpha(0) = q$, there is a unique path $\overline{\alpha} : [0, 1] \to X$ with $\overline{\alpha}(0) = p$ such that $\pi \circ \overline{\alpha} = \alpha$.

   (b) What would it mean for π to have the homotopy lifting property?

   It would mean for each homotopy F between paths α and β starting at q, $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$, there is a unique homotopy $\overline{F}$ between the lifted paths $\overline{\alpha}$ and $\overline{\beta}$ with $\overline{F}(t, 0) = \overline{\alpha}(t)$ and $\overline{F}(t, 1) = \overline{\beta}(t)$ such that $\pi \circ \overline{F} = F$.

   (c) Suppose that $X = \mathbb{R}^2$ and $Y = T^2$ is obtained as the identification space for the equivalence relation $(x, y) \equiv (x + m, y + n)$, for $m, n \in \mathbb{Z}$ on $\mathbb{R}^2$. What is the lift of the diagonal loop in T^2?

   The usual diagonal loop lifts to the path $\alpha(t) = (t, t)$, $0 \leq t \leq 1$. Notice it is not closed.

   (d) Suppose that $\alpha : [0, 1] \to T^2$ is homotopically trival but not constant. Describe its lift to $\mathbb{R}^2$.

   The lift of α will be a closed curve with initial and endpoint $(0, 0)$.

4. Let K be a simplicial complex.

   (a) Define what it means for the realization $|K|$ to be a simplicial manifold of dimension n.

The star of each vertex, which is the realization of the union of the simplices which contain the vertex, is homeomorphic to the closed unit ball in $\mathbb{R}^n$.

(b) Define what it means for the realization $|K|$ to be a simplicial manifold with boundary of dimension n.

The star of each vertex, which is the realization of the union of the simplices which contain the vertex, is homeomorphic to the closed unit ball in $\mathbb{R}^n$, or the half closed unit ball with last coordinate non-negative so that the vertex is mapped to the origin.

(c) Prove the following: If $|K|$ is a simplicial manifold of dimension n and $|L|$ is the simplicial manifold of dimension n with boundary obtained by removing the interior of an n simplex from $|K|$, then $\chi(L) = \chi(K) + (-1)^{n+1}$.

Notice since we are only removing the interior of a simplex, no other simplices are removed besides the single top dimensional simplex. The Euler characteristic is the alternating sums of the number of simplices of each dimension in the complex. Since even dimensional simplices contribute by addition and odd dimensional simplices contribute by subtraction from the total, removing a simplex has the opposite effect. Thus we want to subtract one when the simplex has even dimension and add when it has odd dimension. This is precisely the effect of the formula: $\chi(K) + (-1)^{n+1}$.

5. Let K be a simplicial complex and let $|K|$ be the realization of K.

(a) Define the cone on $|K|$, $C(|K|)$.

The cone on any topological space X is the identification space of the product $X \times [0,1]$ for the equivalence relation $(x,1) \equiv (y,1)$ for any $x, y \in X$.

(b) Construct a simplicial complex CK whose realization $|CK|$ is homeomorphic to $C(|K|)$.

Add a single vertex v_0 to K and for each simplex $\sigma \in K$ of dimension m add the simplex of dimension $m+1$ whose vertices are the $m+1$ vertices of σ and the vertex v_0. Call this simplex σ'. If $|K|$ is a subset of $\mathbb{R}^N$, then $|CK|$ can be represented by the geometric cone on $|K|$ with the cone point $(0,\ldots 0,1) \in \mathbb{R}^{N+1}$.

(c) Prove that $\chi(CK) = 1$. (*Hint:* How should you think about counting simplices of CK?)

Notice the dimensions of σ and σ' are odd and even or even and odd. Consequently, adding σ' has the effect of canceling the contribution of σ to the Euler characteristic. This pairs off all the simplices of CK except for v_0. This leaves an excess 1 in dimension 0, hence $\chi(CK) = 1$.