MATH 363 – Sample First Hour Exam Questions
Selected Solutions

February 24, 2011

The following are solutions to problems that are not definitions or easily available proofs.
From February 22, 2005 (the test had a total of 7 questions):

4. Let $X \subset \mathbb{R}^2$ be defined by

$$X = \{(x, y) : (x+1)^2 + y^2 \leq 1\} \cup \{(x, y) : (x-1)^2 + y^2 < 1\}.$$

(See Figure 1.) Assume that X has the induced topology from $\mathbb{R}^2$. Determine whether the following subsets S of X are open, closed, or neither.

(a) $S = \{(x, y) \in X : (x+1)^2 + y^2 \leq 4\}$.

Closed, since it is the intersection of X with a closed set in the plane.

(b) $S = \{(x, y) \in X : x^2 + y^2 > 1\}$.

Open, since it is the intersection of X with an open set in the plane.

February 23, 2007 (the test had a total of 7 questions):

3. Let $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a function between topological spaces.

(b) Is it possible for a continuous function between topological spaces to be one-to-one and onto, but not be a homeomorphism? If so, give an example, if not, briefly explain why it must be a homeomorphism.

Yes. Consider the identity map, $f(x) = x$, from $(\mathbb{R}, \mathcal{U})$ to $(\mathbb{R}, \mathcal{T}_{FC})$. It is one-to-one, onto and continuous, but the inverse is not continuous.

March 23, 2009 (the test had a total of 6 questions):

1. Consider the plane with the usual topology, $(\mathbb{R}^2, \mathcal{U})$. For each of the following sets $A \subset \mathbb{R}^2$:

- Sketch the set.
- State whether the set is open, closed, or neither and explain your choice.
- Give the closure, interior, and boundary of the set.
- State whether the set is disconnected, connected, or path-connected.

(a) $A = \{(x, y) : |y| < x^2, |x| \leq 1\} \cup \{(0, 0)\}$.

A is the region below the graph of $y = x^2$ and above the graph of $y = -x^2$ (not including the graphs) for x between -1 and 1. It is neither open nor closed. The closure is

$$\overline{A} = \{(x, y) :\ |y| \leq x^2, |x| \leq 1\} \cup \{(0, 0)\}.$$

The interior is

$$A^\circ = \{(x, y) :\ |y| < x^2, |x| < 1\}.$$

The boundary is

$$\partial A\{(x, y) :\ |y| = x^2, |x| \leq 1\} \cup \{(x, y) :\ x = \pm 1, |y| \leq 1\}.$$

It is connected and path connected (though path connected will not be on this test).

(b) $A = \cup_{n=1}^{\infty}\{(x, y) :\ x^2 + y^2 = \frac{1}{n^2}\}$.

A is the infinite union of the collection of circles of radius $\frac{1}{n}$ centered at the origin for $n = 1, 2, \ldots, \infty$. The closure is

$$\overline{A} = A \cup \{(0, 0)\}.$$

The interior is empty. The boundary is the closure. It is disconnected.

2. Let $(X, \mathcal{S})$ be a topological space. Let A and B be subsets of X (not necessarily open or closed).

(a) Prove that $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. (*Hint:* What must you do to show two sets are equal?)

$\Rightarrow$: If $x \in \text{int}(A \cap B)$, there exists an open set $O \subset \text{int}(A \cap B)$ with $x \in O$. Then $x \in O \subset A$ showing $x \in \text{int}(A)$ and $x \in O \subset B$ showing $x \in \text{int}(B)$. Consequently, $x \in \text{int}(A) \cap \text{int}(B)$.
$\Leftarrow$: $\text{int}(A) \cap \text{int}(B)$ is an open set contained in A and B, hence in $A \cap B$. Since $\text{int}(A \cap B)$ is the largest open set contained in $A \cap B$, $\text{int}(A) \cap \text{int}(B) \subset \text{int}(A \cap B)$.

(b) Prove that the boundary of A is a closed set. (*Hint:* Be clear about your definition of boundary and what you must do to show a set is closed.)

The boundary of A is equal to $\overline{A} \cap \overline{A^c}$. This is the intersection of two closes sets, hence is closed.

4. Let $(X, \mathcal{S})$ be a topological space.

(c) Produce a counterexample to show the converse of (b) is false.

Let $X = [-1, 0) \cup (0, 1]$ is disconnected and $Y = (0, 1]$ is connected. $f(x) = x^2$ is onto and continuous providing a counterexample.