MATH 363 – Sample First Hour Exam Questions

February 24, 2011

The following are questions that have appeared on the first hour exams in topology in 2005, 2007, and 2009 that correspond to material we have covered so far this semester. These tests contained other questions we have not covered, just as we have covered some material that was not on those tests.

From February 22, 2005 (the test had a total of 7 questions):

In all the following problems that involve Euclidean space $\mathbb{R}^n$, assume that $\mathbb{R}^n$ has the usual topology. For topological spaces other than $\mathbb{R}^n$, the topology will not be made explicit.

1. Let S be a subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. Complete each of the following definitions:
   (a) $x \in \text{Int}(S)$ if ....
   (b) $x \in \partial(S)$ if ....
   (c) $x \in \overline{S}$ if ....

2. Let $S \subset \mathbb{R}^n$.
   (a) Define what it means for S to be connected.
   (b) Define what it means for S to be compact.

3. Let $f : X \to Y$ be a function between the topological spaces.
   (a) Give the topological definition of continuity for f.
   (b) Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, and define $f : X \to Y$ by $f(x_1, x_2) = x_1$. (The function f is called a *projection*.) Briefly explain why f is continuous. (*Hint*: A carefully explained diagram would suffice.)

4. Let $X \subset \mathbb{R}^2$ be defined by

$$X = \{(x, y) : (x + 1)^2 + y^2 \leq 1\} \cup \{(x, y) : (x - 1)^2 + y^2 < 1\}.$$

   (See Figure 1.) Assume that X has the induced topology from $\mathbb{R}^2$. Determine whether the following subsets S of X are open, closed, or neither.
   (a) $S = \{(x, y) \in X : (x + 1)^2 + y^2 \leq 4\}$.
   (b) $S = \{(x, y) \in X : x^2 + y^2 > 1\}$.

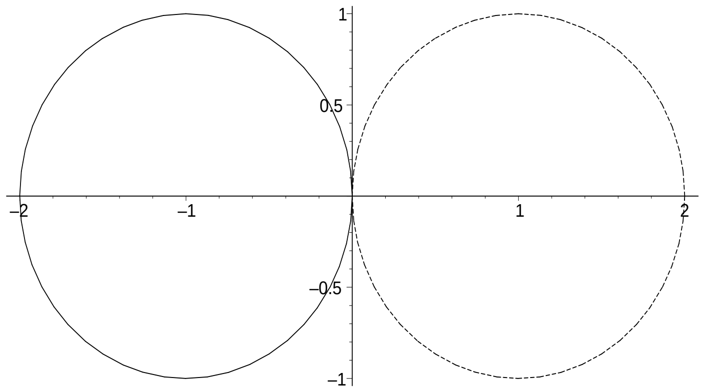

Figure 1:

February 23, 2007 (the test had a total of 7 questions):

In the following, assume that Euclidean space has the usual topology and that any subsets have the subspace topology.

1. Let X be a set and let $\mathcal{T}$ be a collection of subsets of X. What conditions must $\mathcal{T}$ satisfy in order for it to be a *topology* on X?
2. Let $(X, \mathcal{T})$ be a topological space. Let $S \subset X$.
   (a) Define *interior* of S.
   (b) Define *closure* of S.
   (c) Define *boundary* of S.
3. Let $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a function between topological spaces.
   (a) Give the topological definition of continuity for f.
   (b) Is it possible for a continuous function between topological spaces to be one-to-one and onto, but not be a homeomorphism? If so, give an example, if not, briefly explain why it must be a homeomorphism.

March 23, 2009 (the test had a total of 6 questions):

1. Consider the plane with the usual topology, $(\mathbb{R}^2, \mathcal{U})$. For each of the following sets $A \subset \mathbb{R}^2$:

   - Sketch the set.
   - State whether the set is open, closed, or neither and explain your choice.
   - Give the closure, interior, and boundary of the set.
   - State whether the set is disconnected, connected, or path-connected.

   (a) $A = \{(x, y) : |y| < x^2, |x| \leq 1\} \cup \{(0, 0)\}$.

   (b) $A = \cup_{n=1}^{\infty} \{(x, y) : x^2 + y^2 = \frac{1}{n^2}\}$.

2. Let $(X, \mathcal{S})$ be a topological space. Let A and B be subsets of X (not necessarily open or closed).

   (a) Prove that $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$. (*Hint:* What must you do to show two sets are equal?)

   (b) Prove that the boundary of A is a closed set. (*Hint:* Be clear about your definition of boundary and what you must do to show a set is closed.)

3. Let $(X, \mathcal{S})$ be a topological space. (Note: Y is used differently in each part of this problem.)

   (a) Let Y be a subset of X. If $A \subset Y$, what condition must A satisfy in order to be an open set in Y in the subspace topology on Y?

   (b) Let $(Y, \mathcal{T})$ be a second topological space. Let $\mathcal{S} \times \mathcal{T}$ be the product topology on $X \times Y$. If $A \subset X \times Y$, what condition must A satisfy in order to be an open set in $X \times Y$ in the product topology on $X \times Y$?

4. Let $(X, \mathcal{S})$ be a topological space.

   (a) X is connected if what condition holds?

   (b) Let $f : (X, \mathcal{S}) \to (Y, \mathcal{T})$ be an onto continuous function. Prove that if X is connected, then Y is connected.

   (c) Produce a counterexample to show the converse of (b) is false.