Math 363

May 2, 2011

Assignment 11A

This is the first part of a group assignment. It is due Monday, May 9.

In this part of the assignment, you will be calculating the mod 2 homology of some familiar spaces. For the simpler spaces, where the triangulations have relatively few simplices, you can carry out all the details of the calculations. For spaces with a larger number of simplices, you will have to use careful arguments about the structure of the underlying simplicial complexes without listing all their simplices.

1. For each of the following spaces, find a simple triangulation, identify the vector spaces of chains, cycles, and boundaries, and the homology vector spaces. Describe a set of generators for the homology vector spaces spaces. In most cases, you should be able to describe the different vector spaces without listing all the elements.
   (a) A figure 8.
   (b) A figure 8 with two antenna.
   (c) The projective plane.
   (d) A surface of genus g.
2. Let K be a connected simplicial manifold of dimension 2 with boundary.
   (a) Prove that $Z_2(K) = \{0\}$, the trivial additive group.
   (b) If K has a single boundary component, is the boundarys component a cycle? If it is a cycle, is it homologically trivial or not? Explain.
   (c) If K has two disjoint boundary components, are they homologically trivial, not homologically trivial and homologous to each other, or not homologically trivial and homologically distinct?
3. For each of the following spaces, determine the mod 2 homology, including the three dimensional homology. Describe a set of generators for the homology.
   (a) $S^2 \times [-1, 1]$
   (b) $T \times [-1, 1]$.
   (c) T^3.
   (d) S^3.
   (e) $S^2 \times S^1$.

Assignment 11B: Growing Spaces

In the following problems, we consider the question of how the homology of a space changes as the space is being built. The idea is that cycles that appear at one stage of the building process remain cycles throughout the building process, but they may become boundaries, hence homologically trivial as this process continues. Thus the dimensions of the mod 2 homology vector spaces may increase then decrease during this process.

The building process will be captured in what is known as a *filtration* of the simplicial complex or of the topological space. For example for a simplicial complex K, a filtration is a sequence of subcomplexes $K_0 \subset K_1 \subset \ldots K_N = K$ containing an increasing number of simplices of K. Notice that N may or may not be the dimension of the complex K. For a compact topological space X, a filtration is a finite sequence of compact subsets $X_0 \subset X_1 \subset \ldots X_N = X$.

4. This problem is concerned with a complex obtained by modifying an octahedron to obtain a specific complex in $\mathbb{R}^3$. The realization of the complex is shown in Figure 1 and the Maple worksheet distributed with the assignment. Here are the vertices:

$$v_1 = (4,0,0),\ v_2 = (-2,1,0),\ e_v = (-2,-1,0),\ v_4 = (-1,0,2),$$

$$v_5 = (1,-1,0),\ v_6 = (2,1,0), v_7 = (3/2,0,-1),\ v_8 = (2,1,-3).$$

Using the notation e_{ij} for the edge connecting v_i and v_j, the complex K is:

$$\begin{aligned} K \;=\; \{ & v_1, v_2, \ldots, v_8,\ e_{12},\ e_{13},\ e_{14},\ e_{15},\ e_{16},\ e_{23},\ e_{24},\ e_{26},\ e_{28}, \\ & e_{34},\ e_{35},\ e_{38},\ e_{56},\ e_{57},\ e_{58},\ e_{67},\ e_{68},\ e_{78}, \\ & f_{124},\ f_{126},\ f_{135},\ f_{156},\ f_{234},\ f_{238},\ f_{268},\ f_{567},\ f_{578} \} \end{aligned}$$

(Notice the order of the list: after vertices, edges including v_1, then the remaining edges including v_2, etc.)

For each of the following filtrations of K, Determine the homology of each K_i in each dimension. Give simple generators for each mod 2 homology vector space.

(a) Let $K_i = K^i$, the i-skeleton of K, $i = 0, 1, 2$.

(b) Let $K_1 = St(v_1)$, where St denotes the closed star of the vertex, $K_2 = K_1 \cup St(v_2)$, $K_3 = K_2 \cup St(v_3)$, and so on until all of K is included.

(c) For each real number a, let K_a be the union of the simplices of K that lie entirely in the closed half space $x \leq a$. As a increases from $-\infty$, for which values of a does K_a change? Use the subsets K_a for these values to construct the filtration of K.

Can you draw any conclusions about the progression of the homology of subcomplexes of these filtrations as the filtration increases in size? Explain.

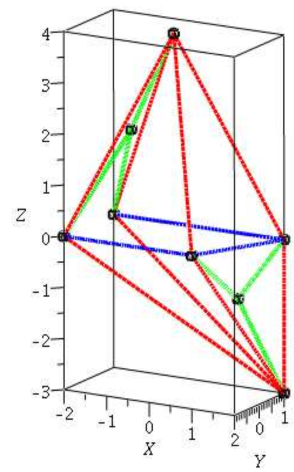

Figure 1: The 1-skeleton for Exercise 4.

5. The attached figure shows a planar data set. Assume that the grid consists of squares of fixed size and all the data points are located at the vertices of the grid. For each real number $\alpha \geq 0$, let K_α denote the simplicial complex produced by the method of α-shapes.
   (a) For which values of α will the number of simplices in K_α jump as α increases? What is the largest gap in these values? (Notice these will either be whole numbers or square roots of whole numbers. Why?)
   (b) On a copy of the data plot, insert, label, and color code edges that appear for the first time for each of the values in (a). Stop the process when the gap between successive values is large. What might this value indicate in terms of the realization of the resulting complex? (Note: Your plot will then have a colored collection of edges that indicate the growth of the 1-skeleton of a complex K.)
   (c) On a second copy of your final plot, color code the faces that are added for each of the values in (a).
   (d) For each of the values from (a), what are the mod 2 homology vector spaces of the complexes K_α.
   (e) For which K_α does it appear that the homology reflects the homology of the underlying space from which the data was sampled? Explain. (Does this have anything to do with (a)?)