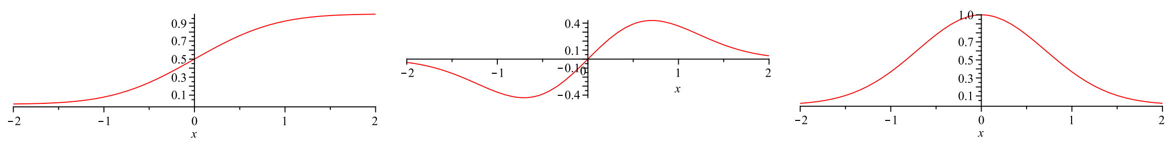
MATH 134
Second Hour Exam Sample Solutions
April 3, 2011

You may use your calculator and integral tables. Indicate any calculations you do with the calculator, indicate which formula you use from the tables and the values of any constants that appear in the formula, and show your algebra whenever calculations are done by hand.

1. (20 pts.) Short answer questions:

   (a) Of the following three graphs, which could be the graph of a probability density function, which could be the graph of a cumulative distribution function, which is neither? Briefly explain your choices.



(a) (b) (c)

Figure 1:

Since probability density functions and cumulative distributions are non-negative, (b) is neither. Since (a) is increasing with asymptotes at 0 for $-\infty$ and 1 for ∞, (a) is a cumulative distribution function. This leaves (c) for a density function.

   (b) Suppose that $y = f(x)$ and $y = g(x)$ satisfy $f(x) \geq g(x) \geq 0$ on $1 \leq x \leq \infty$. Which of the following is true? Explain briefly.

      i. If $\int_1^\infty f(x)\,dx$ converges then $\int_1^\infty g(x)\,dx$ converges.

      ii. If $\int_1^\infty g(x)\,dx$ converges then $\int_1^\infty f(x)\,dx$ converges.

   The first statement is true, because for any interval $a \leq x \leq b$, $f(x) \geq g(x) \geq 0$ implies that $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \geq 0$. This is preserved taking the limit as $b \to \infty$. So if $\int_1^\infty f(x)\,dx$ is a finite value, $\int_1^\infty g(x)\,dx$ must also be finite.

2. (20 pts.) The region A in the plane is shown in Figure 4. It lies between the graphs of $x = -y^2 - 1$, $y = x$, and $y = 0$ and $y = 1$.

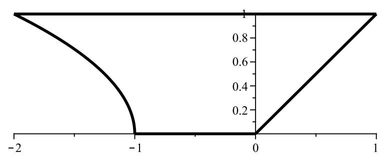


Figure 2:

(a) Label the coordinates of the corners of A on the plot.

The upper left corner is $(-2, 1)$, the upper right corner is $(1, 1)$, the lower left corner is $(-1, 0)$, and the lower right corner is $(0, 0)$.

(b) Compute the area of A.

The simpler approach is to evaluate an integral with respect to y between the graphs of $x = y$ and $x = -y^2 - 1$.

$$\begin{aligned}
\int_0^1 y - (-y^2 - 1)\, dy &= \int_0^1 y + y^2 + 1\, dy \\
&= \left[\frac{1}{2}y^2 + \frac{1}{3}y^3 + y\right]_0^1 \\
&= \frac{1}{2} + \frac{1}{3} + 1 \\
&= \frac{11}{6}
\end{aligned}$$

3. (20 pts.) The region A in the plane is defined by the following inequalities:

$$0 \leq x \leq y^2, \qquad \text{and} \qquad -1 \leq y \leq 1.$$

(a) Sketch the region A.

See Figure 3(a).

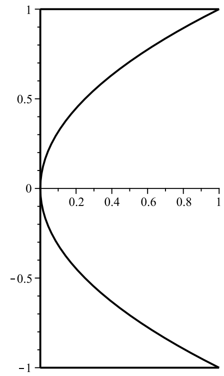
(b) Sketch the solid of revolution obtained by rotating the region about the line $x = 2$.

See Figure 3(b).

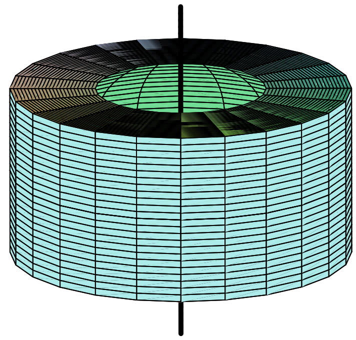
(c) Find the volume of the region from part (b).

Using the method of slices for a vertical axis.

$$\begin{aligned}
\pi \int_{-1}^1 2^2 - (2 - y^2)^2\, dy &= \pi \int_{-1}^1 4y^2 - y^4\, dy \\
&= \pi \left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]_{-1}^1 \\
&= \frac{34}{15}\pi \approx 7.12.
\end{aligned}$$



(a)



(b)

Figure 3:

Using the method of shells for a vertical axis, we integrate over the top half of the region and double the result.

$$\begin{aligned} 2\cdot 2\pi \int_0^1 (2-x)(1-\sqrt{x})\,dx &= 4\pi \int_0^1 2 - 2\sqrt{x} - x + x^{3/2}\,dx \\ &= 4\pi \left[2x - \frac{4}{3}x^{3/2} - \frac{1}{2}x^2 + \frac{2}{5}x^{5/2}\right]_0^1 \\ &= 4\pi(2 - \frac{4}{3} - \frac{1}{2} + \frac{2}{5}) \\ &= \frac{34}{15} \approx 7.12. \end{aligned}$$

4. (20 pts.) Consider the triangle with vertices $(0,0)$, $(4,2)$, and $(4,0)$. Find the coordinates of the center of mass of the triangle.

The area of the triangle is 4, so we need only compute two integrals:

$$\begin{aligned} \overline{x} &= \frac{1}{4}\int_0^4 x(\frac{1}{2}x)\,dx = \frac{1}{4}\left[\frac{1}{6}x^3\right]_0^4 = \frac{8}{3}. \\ \overline{y} &= \frac{1}{4}\int_0^4 \frac{1}{2}(\frac{1}{2}x)^2\,dx = \frac{1}{4}\left[\frac{1}{24}x^3\right]_0^4 = \frac{2}{3} \end{aligned}$$

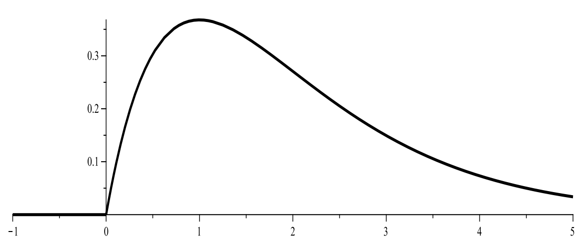


Figure 4:

5. (20 pts.) The above figure shows the graph of the function

$$f(x) = \begin{cases} 0 & x < 0 \\ xe^{-x} & x \geq 0 \end{cases}$$

(a) Show that $f(x)$ is a probability density function.

Since the function is non-negative, we need only show the integral over entire line is equal to 1.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_0^{\infty} xe^{-x}\,dx \\ &= \lim_{b\to\infty} \int_0^b xe^{-x}\,dx \\ &= \lim_{b\to\infty} \left[-xe^{-x} + \int e^{-x}\,dx\right]_0^b \\ &= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_0^b \\ &= \lim_{b\to\infty} \left[-be^{-b} - e^{-b}\right] - (-1) \\ &= 1. \end{aligned}$$

Notice that we used the fact that $\lim_{b\to\infty} be^{-b} = 0$. This can be shown by using l'Hopital's rule for the indeterminate form $\infty \cdot 0$.

(b) Find the mean of the density function f.

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) &= \int_0^{\infty} x^2e^{-x}\,dx \\ &= \lim_{b\to\infty} \int_0^b x^2e^{-x}\,dx \\ &= \lim_{b\to\infty} \left[-x^2e^{-x} + 2\int xe^{-x}\,dx\right]_0^b \\ &= \lim_{b\to\infty} \left[-xe^{-x} + 2(-xe^{-x} - e^{-x})\right]_0^b \\ &= \lim_{b\to\infty} \left[-be^{-b} + 2(-be^{-b} - e^{-b})\right] - 2(-1) \\ &= 2. \end{aligned}$$

Notice that we used the integral for part (a) to do the second integration by parts.

(c) Is the median of f larger than the mean or is the mean larger than the median? Explain your answer. (*Hint:* This question can be answered based on the shape of the density and WITHOUT calculations.)

The median will be less than 2. This is because the long right tail has the effect of pulling the mean to the right of the median.