Math 134

Final Exam Sample Solutions
May 6, 2011

You may use your calculator and integral tables. Show all your work in the blue book. If you use a formula from the tables, state which formula and the values of any constants in the formula.

1. (15 pts.) Let F be defined by the integral:

$$F(x) = \int_0^x \frac{1}{1+t^4}\,dt$$

   (a) What is $F'(x)$?

   By the first fundamental theorem of calculus, the $F'(x) = \frac{1}{1+x^4}$.

   (b) Which of the following is correct, explain:

   - F is always increasing.
   - F is always decreasing.
   - F is increasing for some values of x and decreasing for others.

   Since $F'(x) = \frac{1}{1+x^4}$ is positive for all x, F is always increasing.

   (c) Find the interval(s) on which F is concave up and the interval(s) on which F is concave down.

   Since $F''(x) = -\frac{4x^3}{(1+x^4)^2}$, $F''(x) > 0$ for $x < 0$ and $F''(x) < 0$ for $x > 0$. Thus F is concave up for $x < 0$ and concave down for $x > 0$.

2. (15 pts.) Let $f(x) = \sin(\pi x^2)$. Let L_4 denote the left endpoint approximation to $f(x)$ on the interval $[a, b] = [0, 1]$ that uses 4 rectangles.

   (a) Sketch the rectangles for L_4 on a graph of f on the interval $[0, 1]$.

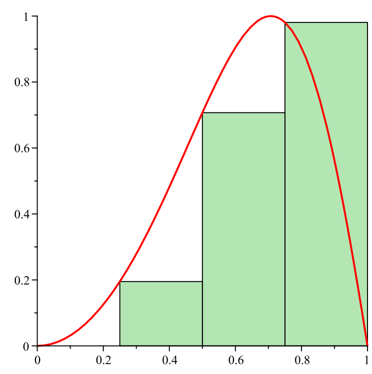

Figure 1:

   (b) Calculate L_4. Be sure to show the expanded form of your sum before evaluating the sum. Your final answer should be a number.

$$\sum_{i=0}^{3} f(i/4)\frac{1}{4} = 0 \cdot \frac{1}{4} + \sin(\pi(1/4)^2) \cdot \frac{1}{4} + \sin(\pi(2/4)^2) \cdot \frac{1}{4} + \sin(\pi(3/4)^2) \cdot \frac{1}{4} \approx 0.539.$$

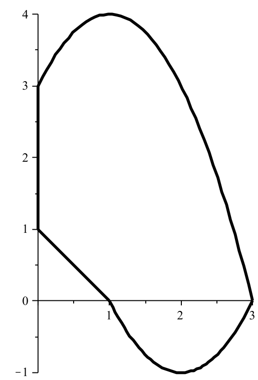

Figure 2:

3. (14 pts.)Let A be the region shown in Figure 2. It lies

- above the segment $y = 1 - x$ and the curve $y = x^2 - 4x + 3$,
- below the curve $y = 3 + 2x - x^2$, and
- to the right of the y-axis.

Find the area of A.

The region is naturally set up to integrate with respect to x. Since the lower curve has a split definition, it is necessary to split the region at $x = 1$ and do two integrals.

$$\begin{aligned}
& \int_0^1 (3 + 2x - x^2) - (1 - x)\,dx + \int_1^3 (3 + 2x - x^2) - (x^2 - 4x + 3)\,dx \\
= & \int_0^1 2 + 3x - x^2\,dx + \int_1^3 6x - 2x^2\,dx \\
= & \left[2x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^1 + \left[3x^2 - \frac{2}{3}x^3\right]_1^3 \\
= & 2 + \frac{3}{2} - \frac{1}{3} + (27 - 18) - (3 - \frac{2}{3}) \\
= & 9\frac{5}{6}
\end{aligned}$$

4. (36 pts.) Evaluate the following indefinite and definite integrals.

(a) Use a substitution $u = \cos(s)$ or rewrite in terms of tan and sec.

$$\int \frac{\sin(s)}{\cos^2(s)}\,ds = \cos^{-1}(s) + C, \text{ or } = \int \tan(s)\sec(s)\,ds = \sec(s) + C.$$

(b) Using integration by parts with $u = \ln(t)$ and $dv = t^3 dt$:

$$\begin{aligned} \int_1^e t^3 \ln(t)\,dt &= \frac{1}{4}t^4 \ln(t) - \int \frac{1}{4}t^4 \frac{1}{t}\,dt \\ &= \frac{1}{4}t^4 \ln(t) - \int \frac{1}{4}t^3\,dt \\ &= \frac{1}{4}t^4 \ln(t) - \frac{1}{20}t^4 + C. \end{aligned}$$

(c) Use partial fractions and solve the following for A, B, and C. :

$$\frac{3x^2 + x + 4}{(x-2)(x^2+2)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+2}$$

Choose convenient values for x to solve for the coefficients:

$$x = 2 \rightarrow 18 = A \cdot 6 + 0 \rightarrow A = 3.$$

With $A = 3$:

$$x = 0 \rightarrow 4 = 3 \cdot 2 + C(-2) \rightarrow C = 1.$$

With $A = 3$ and $C = 1$:

$$x = 1 \rightarrow 8 = 3 \cdot 3 + (B+1)(-1) \rightarrow B = 0.$$

This allows us to rewrite the integrand:

$$\begin{aligned} \int \frac{3x^2 + x + 4}{(x-2)(x^2+2)}\,dx &= \int \frac{3}{x-2} + \frac{1}{x^2+2}\,dx \\ &= 3\ln|x-2| + \frac{1}{\sqrt{2}}\arctan(\frac{1}{\sqrt{2}}x) + C \end{aligned}$$

5. (20 pts.) Let A be the region in the plane lying below the graph of $y = \sin(x)$ and above the x-axis for $0 \leq x \leq \pi$.

(a) Sketch the region A.

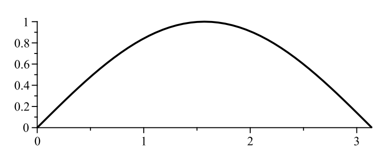

Figure 3:

(b) Find the volume of the solid obtained by rotating A about the x-axis. (*Hint:* Which method should you use?)

Use the method of slices:

$$\begin{aligned} \int_0^{\pi} \pi \sin^2(x)\, dx &= \pi \int_0^{\pi} \frac{1}{2}(1 - \cos(2x))\, dx \\ &= \pi \left[\frac{1}{2}(x - \frac{1}{2}\sin(2x))\right]_0^{\pi} = \frac{\pi^2}{2} \end{aligned}$$

(c) Find the volume of the solid obtained by rotating A about the line $x = -2$. (*Hint:* Which method should you use?)

Use the method of shells and then parts:

$$\begin{aligned} \int_0^{\pi} 2\pi \sin(x)(x+2)\, dx &= 2\pi \int_0^{\pi} x\sin(x) + 2\sin(x)\, dx \\ &= 2\pi \left[-x\cos(x) + \int \cos(x)\, dx\right]_0^{\pi} + [-2\cos(x)]_0^{\pi} \\ &= 2\pi \left[-x\cos(x) + \sin(x) - 2\cos(x)\right]_0^{\pi} \\ &= 2\pi^2 + 8\pi. \end{aligned}$$

6. (20 pts.) (Hypothetical) The time to failure of a certain popular brand of notebook computer is modeled by an exponential density function with mean 4 years.

(a) Write out the formula for this function.

$$f(x) = \frac{1}{4}e^{-x/4}$$

(b) If 400 Holy Cross first-year students purchase one of these computers when they arrive on campus in August, how many can expect to have working computers in August of their sophomore year?

To find the percentage with working computers, first compute the integral of f from 0 to 1 to determine the percentage whose computers failed.

$$\int_0^1 \frac{1}{4}e^{-x/4}\, dx = \left[-e^{-x/4}\right]_0^1 = 1 - e^{1/4} \approx 0.222.$$

This tells us that 22.2% will have failed, so 77.8% of 400 or approximately 311 will be working.

(c) Will half of these 400 computers fail by the time these students graduate (which is 45 months after they entered)?

$$\int_0^{3.75} \frac{1}{4}e^{-x/4}\, dx = \left[-e^{-x/4}\right]_0^{3.75} = 1 - e^{-3.75/4} \approx 0.608.$$

So approximately 60.8% will have failed, which is over half.

7. (20 pts.) Consider the differential equation:

$$\frac{dP}{dt} = 2(5 - P).$$

(a) Find the general solution to the equation.

Separate variables, then integrate to find P.

$$\frac{dP}{P-5} = -2\,dt \rightarrow \int \frac{dP}{P-5} = -2 \int dt \rightarrow \ln|P-5| = -2t + C$$

$$\rightarrow |P-5| = Ae^{-2t} \rightarrow P = Ae^{-2t} + 5.$$

Here we assumed that $P > 5$ to remove the absolute value sign, which is consistent with part (b).

(b) Find the particular solution that satisfies $P(0) = 10$.

$$10 = Ae^0 + 5 \rightarrow A = 5 \rightarrow P(t) = 5e^{-2t} + 5.$$

8. (20 pts.) Wallabies are marsupials (the young mature in an external pouch on the mother's stomach) related to kangaroos. There is a small colony of wallabies on the Isle of Man off the coast of England. (All true so far.) Consider the following logistic differential equation model for this population P.

$$\frac{dP}{dt} = .01 \cdot P(100 - P).$$

(a) What is the carrying capacity for this model?

The carrying capacity is 100.

(b) For which values of P will the population increase? For which values of P will the population decrease?

For $P < 100$ $dP/dt > 0$, so the population will increase for $P < 100$. For $P > 100$ $dP/dt < 0$, so the population will decrease for $P > 100$.

(c) Sketch the line field for this model on the rectangle $0 \le t \le 100$ and $0 \le P < 150$. (*Hint:* It suffices to sketch the lines when $P = 25$, 50, 75, 100, and 125.)

(d) Based on your plot, sketch a solution curve to the equation for the initial condition $P(0) = 25$.

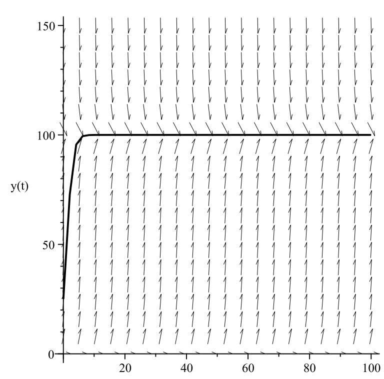

Figure 4:

9. (20 pts.)

(a) Find the following limit:

$$\lim_{n\to\infty} \frac{n^2+(-1)^n}{(-n)^3+1} = \lim_{n\to\infty} \frac{1+(-1)^n\frac{1}{n^2}}{-n+\frac{1}{n^2}} = 0$$

(b) Find the sum of the following series:

This is a geometric series with $r = \frac{1}{4}$ and $A = \frac{3^2}{4^2}$.

$$\sum_{n=2}^{\infty} \frac{3^2}{4^n} = \frac{3^2}{4^2}\frac{1}{1-\frac{1}{4}} = \frac{3^2}{4^2}\frac{4}{3} = \frac{3}{4}.$$

(c) Does the following series converge or diverge? (*Hint:* What test should you use?)

$$\sum_{n=1}^{\infty} \frac{n+1000}{n^3+1}$$

Use a limit comparison test comparing the series to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is p series with $p = 2$.

$$\lim_{n\to\infty} \frac{\frac{n+1000}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+1000n^2}{n^3+1} = 1$$

applying l'Hopital's rule. Since the limit is non-zero, the series have the same convergence behavior. We conclude that the original series converges.

10. (20 pts.) Consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{4^n n^2}$$

(a) Find the radius of convergence of the series.

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{4^{n+1}(n+1)^2}\right|}{\left|\frac{x^n}{4^n n^2}\right|} = \lim_{n\to\infty} \left|\frac{x\cdot n^2}{4(n+1)^2}\right| = |x|/4.$$

The series will converge for $|x|/4 < 1$ or $|x| < 4$, so the radius of convergence is 4.

(b) Find the interval of convergence of the series.

Check whether the series converges for 4 and -4. For $x = 4$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a convergent p series with $p = 2$. For $x = -4$,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

This is a convergent alternating series. Combining these we have that the interval of convergence is $[-4, 4]$.

(c) Find the derivative of this series.

$$\sum_{n=1}^{\infty} n\frac{x^{n-1}}{4^n n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{4^n n}$$

11. (20 pts.)

(a) What is the MacLaurin series for the function $y = e^x$?

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(b) Use your answer to (a) to write a series for $y = e^{-x}$.

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

(c) Use your answers to (a) and (b) to write a series for the function $C(x) = \frac{1}{2}(e^x + e^{-x})$.

Adding the two series, the odd terms will cancel leaving only twice the even terms:

$$\begin{aligned} \frac{1}{2}(e^x + e^{-x}) &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{x^n}{n!} + (-1)^n \frac{x^n}{n!} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} 2\frac{x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \end{aligned}$$

(d) Based on the terms that appear in your series for $C(x)$, what sort of symmetry does $C(x)$ have? Explain.

Since only even terms appear, the function has even symmetry.