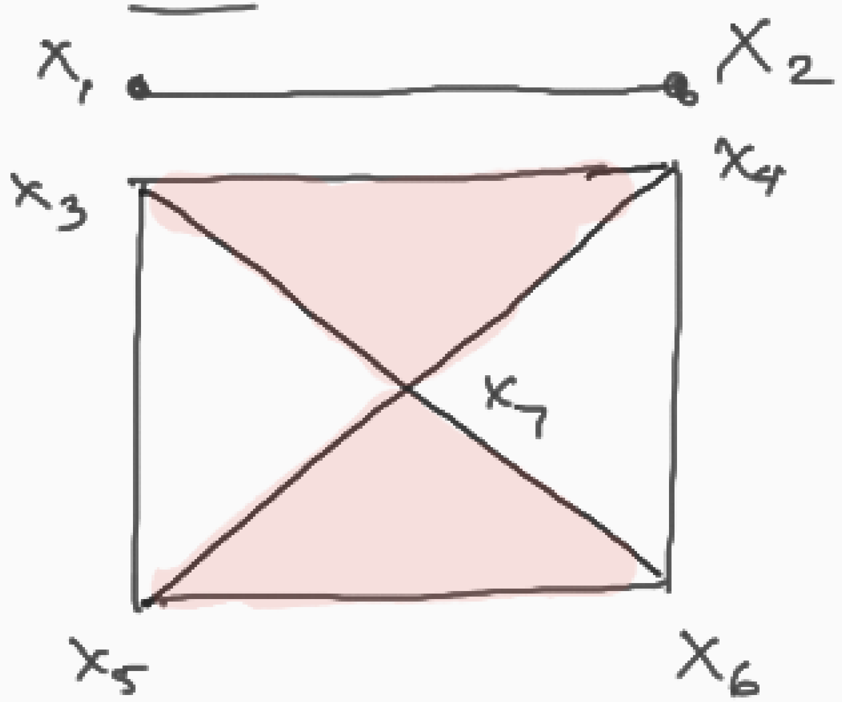
3/21/20 Topology Review Part C: Computing Simplicial Homology

- Simplicial homology vector spaces (or groups) are quotients Z_i/B_i of cycles mod boundaries
- Calculating Z_i/B_i has three parts
  - First, calculate $Z_i = \mathrm{Ker}(\partial_i)$, this means find a basis for Z_i
  - Second, calculate $B_i = \mathrm{Im}(\partial_{i+1})$, this means find a basis for B_i
  - Third, determine equivalence classes of cycles that form a basis for Z_i/B_i

To carry out the calculations, work with the matrices of the boundary transformations with respect to the bases for $C_0(X)$, $C_1(X)$, ...

Ex.



Bases: (order matters)

$C_0(X) : \{x_1, x_2, \dots x_7\} = \alpha$

$C_1(X) : \{e_{12}, e_{34}, e_{35}, e_{37}, e_{46}, e_{47}, e_{56}, e_{57}, e_{67}\} = \beta$

$C_2(X) : \{f_{347}, f_{567}\} = \gamma$

$$0 \xrightarrow[\partial_3]{} C_2(X) \xrightarrow[\partial_2]{} C_1(X) \xrightarrow[\partial_1]{} C_0(X) \xrightarrow[\partial_0]{} 0$$

Calculating $H_2 = Z_2 / B_2$

Since $B_2 = \mathrm{Im}(\partial_3) = \{0\}$, $H_2 = Z_2 \Rightarrow$ Only need to find $\mathrm{Ker}(\partial_2)$

$\beta = \{e_{12}, \ldots, e_{67}\}$, $\gamma = \{f_{347}, f_{567}\}$

$$[\partial_2]_\gamma^\beta = \begin{array}{c} \\ e_{12} \\ e_{34} \\ e_{35} \\ e_{37} \\ e_{46} \\ e_{47} \\ e_{56} \\ e_{57} \\ e_{67} \end{array} \begin{array}{c} \begin{array}{cc} f_{347} & f_{567} \end{array} \\ \begin{bmatrix} & \\ 1 & \\ & \\ 1 & \\ & \\ 1 & \\ & 1 \\ & 1 \\ & 1 \end{bmatrix} \end{array} \longrightarrow \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

2 Basic Variables

0 Free Variables

$\dim(\mathrm{Ker}(\partial_2)) = 0$

$H_2(X) = \{0\}$

$$0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

Calculating $H_1 = Z_1 / B_1$

From previous page, $B_1 = \text{span}\{e_{34}+e_{37}+e_{47},\ e_{56}+e_{57}+e_{67}\}$

columns of basic variables

$\beta = \{e_{12}, \dots, e_{67}\}$ $\qquad \alpha = \{x_1, \dots, x_7\}$

$$[\partial_1]_\beta^\alpha = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_{12} & e_{34} & e_{35} & e_{37} & e_{46} & e_{47} & e_{56} & e_{57} & e_{67} \end{array} \\ \begin{bmatrix} 1 & & & & & & & & \\ 1 & & & & & & & & \\ & 1 & 1 & 1 & & & & & \\ & 1 & & & 1 & 1 & & & \\ & & 1 & & & & 1 & 1 & \\ & & & & 1 & & 1 & & 1 \\ & & & 1 & & 1 & & 1 & 1 \end{bmatrix} \end{array}$$

Notes (1) Ordering edges $\Rightarrow$ simple matrix form

(2) Since addition is subtraction mod 2 addition is the only operation needed

$$0 \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$$

Calculating $H = Z_1 / B_1$

Row Operations $\Rightarrow$

$$[\partial_1]_\beta^\alpha = \begin{array}{c} \begin{array}{ccccccccc} \multicolumn{5}{c}{\text{basic}} & \multicolumn{4}{c}{\text{free}} \\ e_{12} & e_{34} & e_{35} & e_{37} & e_{46} & e_{47} & e_{56} & e_{57} & e_{67} \end{array} \\ \begin{bmatrix} 1 & & & & & & & & \\ & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ & & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ & & & 1 & 0 & 1 & 0 & 1 & 1 \\ & & & & 1 & 0 & 1 & 0 & 1 \\ & & & & & & & & \end{bmatrix} \end{array}$$

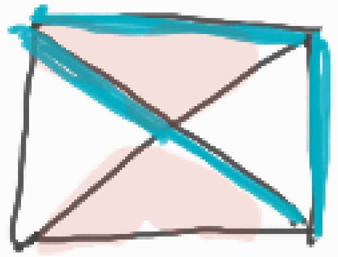
Set free variables alternately $= 1$

e.g. coefficient of $e_{67} = 1 \Rightarrow$

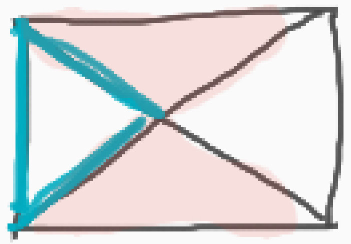
$1 \cdot e_{34} + 1 \cdot e_{37} + 1 \cdot e_{46} + 1 \cdot e_{67} = 0$

So $e_{34} + e_{37} + e_{46} + e_{67}$ is a basis vector for $\mathrm{Ker}(\partial_1)$

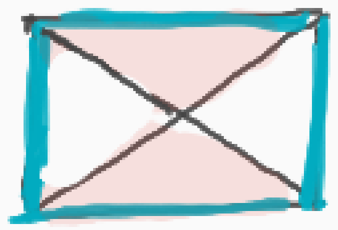
Basis:



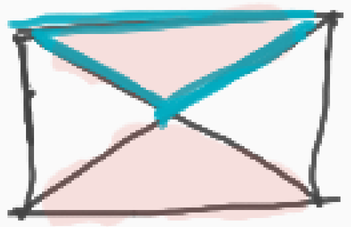
$e_{34} + e_{37} + e_{46} + e_{67}$



$e_{35} + e_{37} + e_{57}$



$e_{34} + e_{35} + e_{46} + e_{56}$



$e_{34} + e_{37} + e_{47}$

Observations:

$z_4 = \partial_2(f_{347})$, $\quad z_1 + z_2 + z_3 = \partial_2(f_{567})$

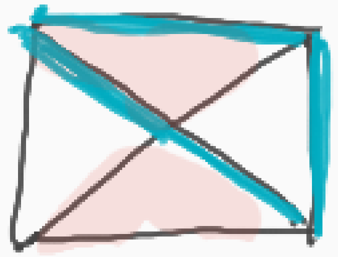
z_2 "captures" the tunnel on the left

$z_1 + z_4$ "captures" the tunnel on the right

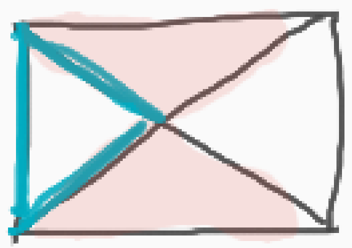
Homology: $[0]$, $[z_2]$, $[z_1+z_4]$, $[z_1+z_2+z_4]$

Notice: $z_1 \in [z_1+z_4]$ since $(z_1+z_4)-z_1 = z_4 = \partial_2(f_{347})$;
$z_4 \in [0]$ since $z_4 = \partial_2(f_{347})$; $z_3 \in [z_1+z_2+z_4]$ since $z_1+\dots+z_4 = \partial_2(f_{347}+f_{567})$

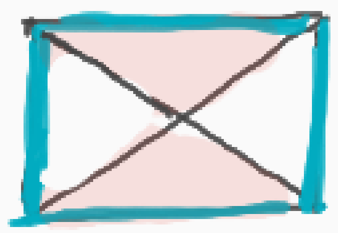
Basis:

$z_1 =$ 

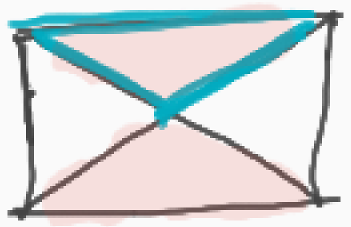
$e_{34}+e_{37}+e_{46}+e_{67}$

$z_2 =$ 

$e_{35}+e_{37}+e_{57}$

$z_3 =$ 

$e_{34}+e_{35}+e_{46}+e_{56}$

$z_4 =$ 

$e_{34}+e_{37}+e_{47}$

Summary:

For this example, we've shown

$$H_2(X) = \{0\}$$

$$H_1(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$\cong \{[0], [z_2], [z_1 + z_4], [z_1 + z_2 + z_4]\}$$

A further calculation would show

$$H_0(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$$= \{[0], [x_1], [x_3], [x_1 + x_3]\}$$