Persistence Barcodes to Persistence Diagrams

The information in a persistence bar code or just barcode is about the persistence of topological features across a filtration of a simplicial complex X:

$$\underline{X}_1 \subset \underline{X}_2 \subset \underline{X}_3 \subset \cdots \subset \underline{X}_n = X$$

We calculate homology $H_j(X_i) \leftarrow$

cycle dimension ↗ ↖ filtration index

and

$$In_* : H_j(\underline{X_i}) \to H_j(\underline{X_{i+1}}) \quad \text{for} \quad In : \underline{X_i} \to \underline{X_{i+1}}$$

This leads to In_* Induced maps In_*

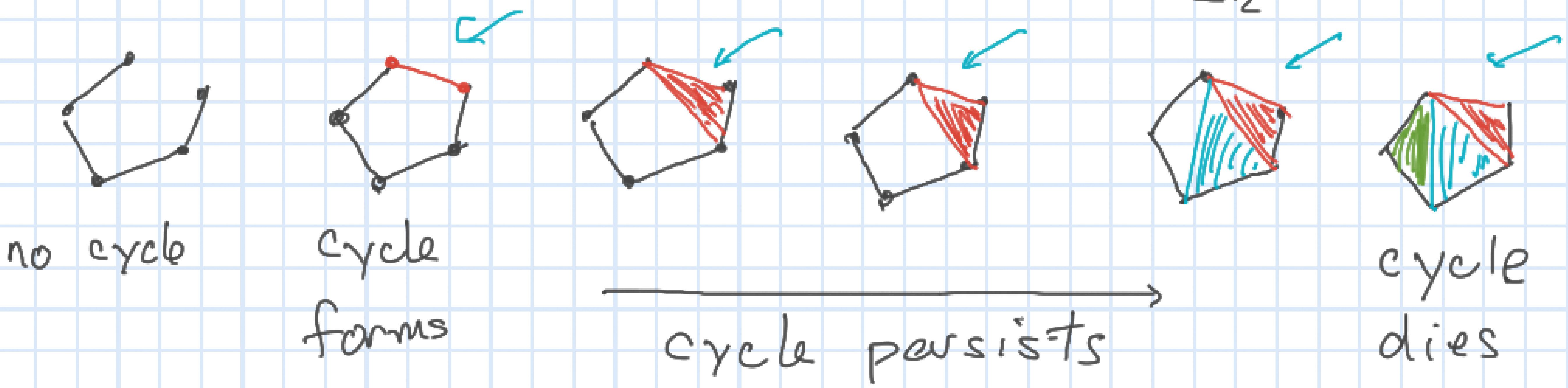
$$\to H_j(\underline{X_{i-1}}) \xrightarrow{} H_j(\underline{X_i}) \to \cdots \to H_j(\underline{X_{k-1}}) \to H_j(\underline{X_k}) \to \cdots$$

- Each homology vector space $= \underline{\underline{\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2}}$ for some # of copies.
- Each copy of $\underline{\mathbb{Z}_2}$ represents a cycle.
- Sequence decomposes into sums of

$$\underline{0} \to \underset{\substack{\uparrow \\ \text{birth time}}}{\underline{\mathbb{Z}_2}} \to \cdots \to \underline{\mathbb{Z}_2} \to \underset{\substack{\uparrow \\ \text{death time}}}{\underline{0}}$$

Schematic decomposition in dimension 1

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$$



- Sequence of homology groups is made up of sequences like this.
- These $\mathbb{Z}_2$ sequences are essentially unique

Each bar in a barcode represents a

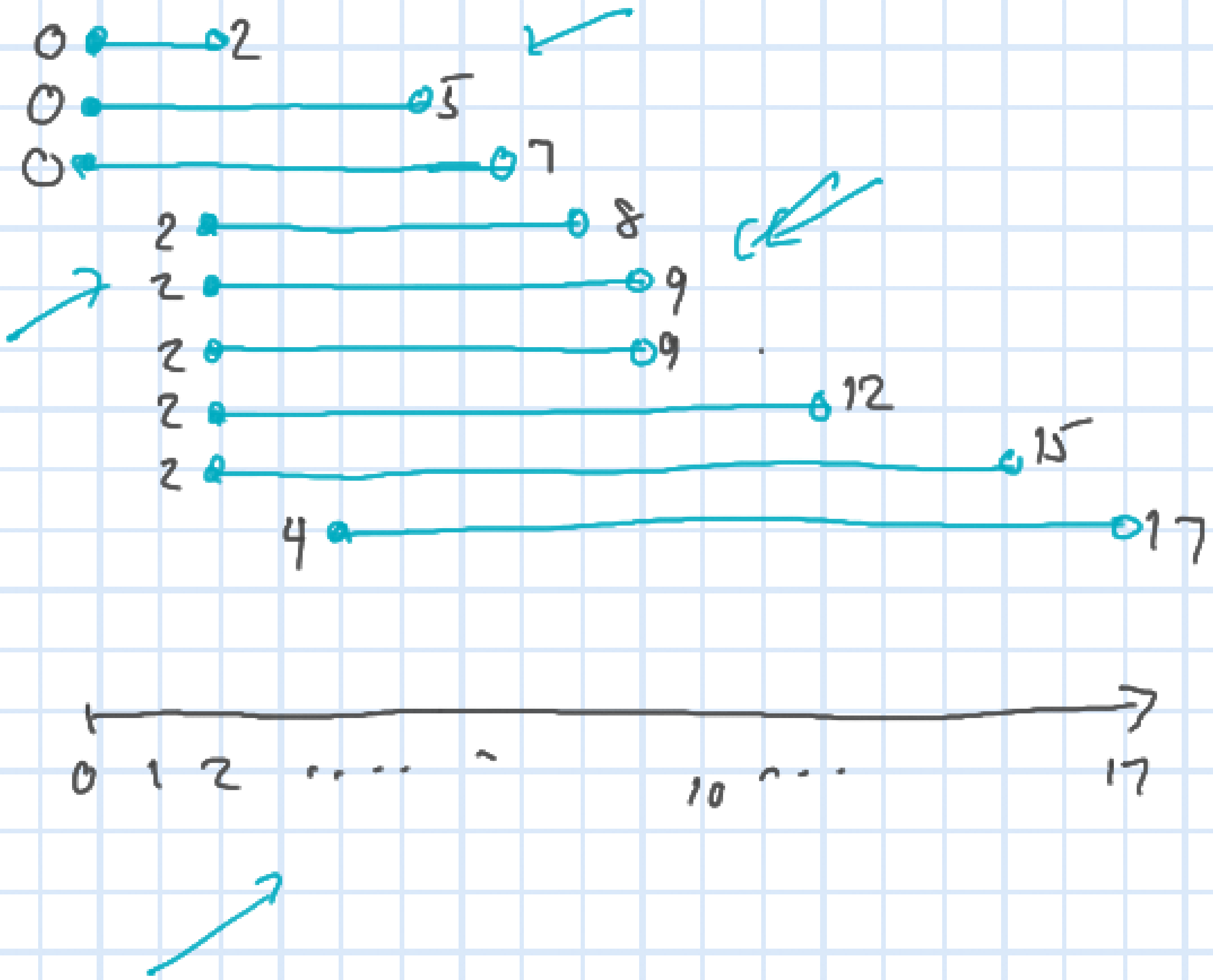
$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{Z}_2 \arrow[r] & \cdots \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d, phantom, "\cap"] & 0 \arrow[d, phantom, "\cap"] \\
 & H(X_b) & & & H(X_{d-1}) & H(X_d)
\end{tikzcd}$$

Birth time b, death time d.

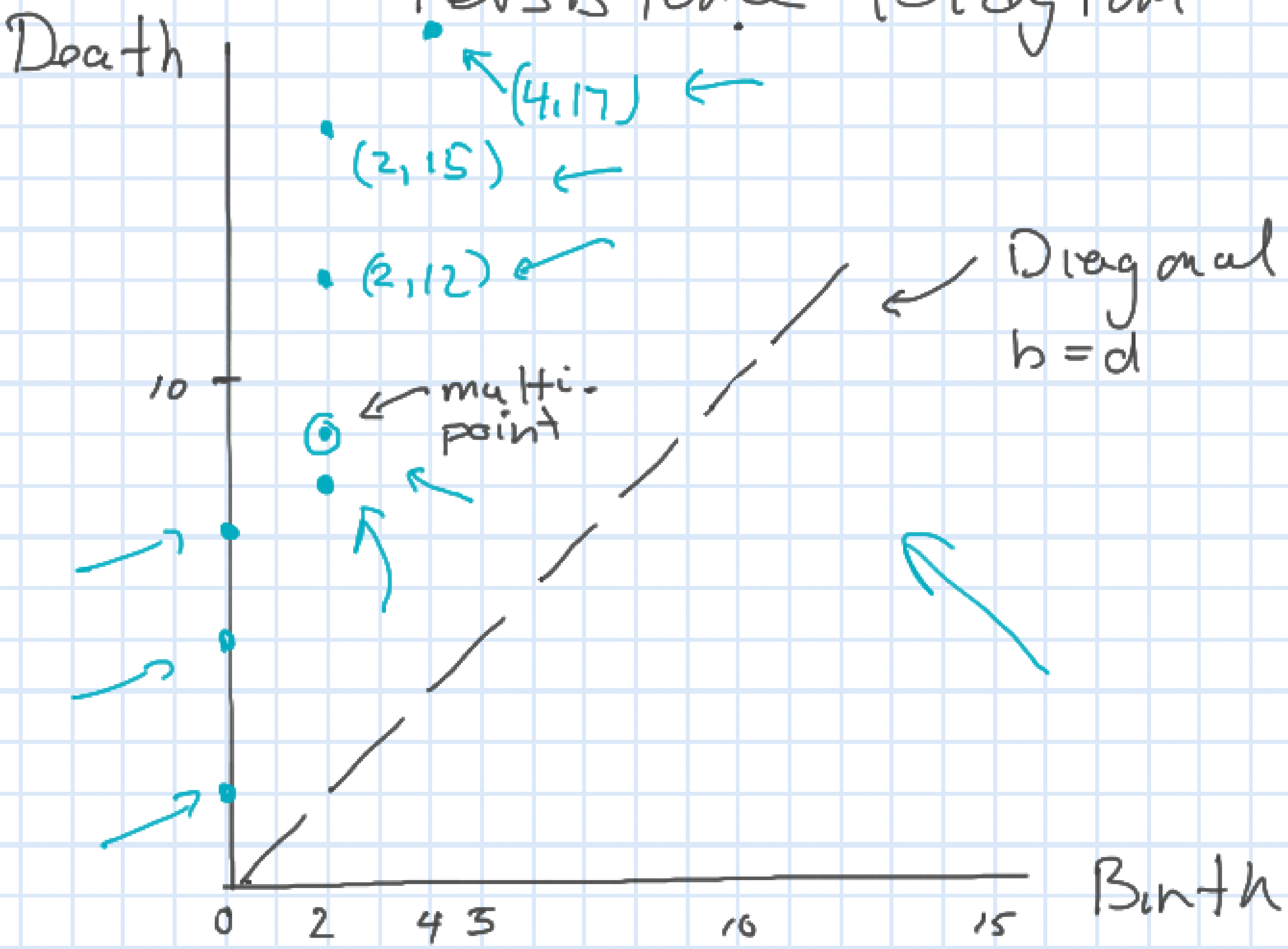
- Barcodes summarize this algebraic information
- Barcodes are unwieldy, awkward.

Persistent diagrams are an alternative to barcodes but contain the same information

Bar Code



Persistence Diagram



Observations:

- If the filtrations are indexed by an increasing index, all persistence points lie above diagonal So $d \geq b$.
- Persistance points have multiplicity
- Cycles born at time 0 have points on d-axis
- Large lifetimes $d-b$ identified with significant features.
- Short lifetimes $d-b$ interpreted (usually) as topological noise. near diagonal