Introduction to Persistent Homology

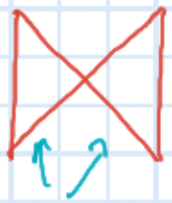
<u>Persistent</u> <u>homology</u> applies the homology construction to the subcomplexes of a filtration of a simplicial complex.

The starting point is the <u>induced</u> <u>map</u> on homology for the inclusion of one complex in another. The next two slides review this concept.

Intuition: If X is a simplicial complex and Y is a subcomplex, then every cycle in Y (of any dimension) is also a cycle in X.

This is because the property of being a cycle is about how the simplices in the cycle fit together.

Example: z is a cycle no matter where A is located

$z =$ 

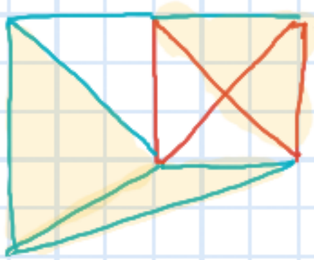
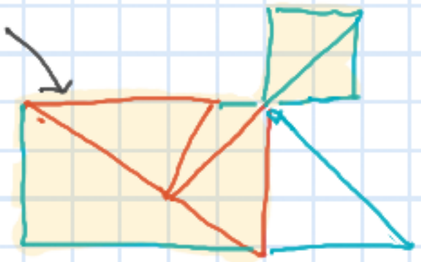


image of z



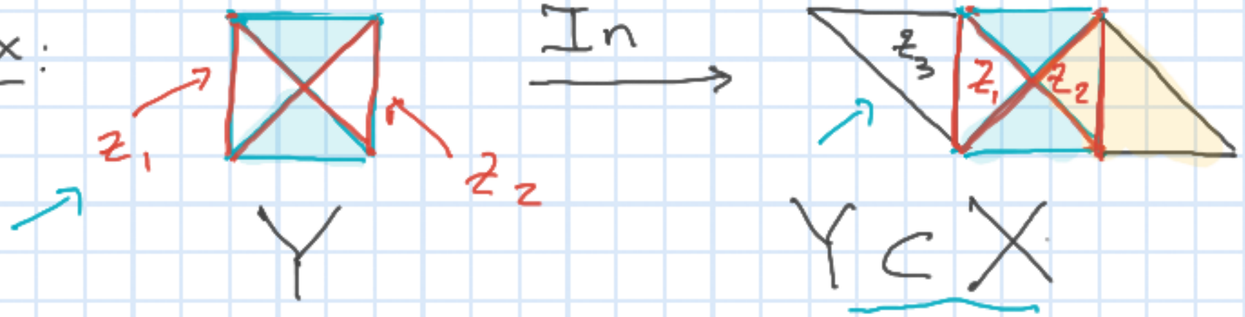
z captures 1 tunnel

z captures 0 tunnels

$\partial z = 0$ in each case

Suppose Y is a subcomplex of X. Denote the simplicial map that maps each simplex of Y to itself in X by $In: Y \to X$.

Ex:



$$H_1(Y) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{In_*} H_1(X) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

$[z_1]$ $[z_2]$ $\qquad$ $[z_1]$ $[z_3]$

Induced Map

$[z_1] \longrightarrow [z_1]$

$[z_2] \longrightarrow [0]$ (a boundary)

$[z_3]$

Cycles in Y can become boundaries

X can contain cycles not in Y

If Y is a subcomplex of X, $In : Y \to X$, we have induced maps In_* in all dimensions

		maps
$\to$	$In_* : H_0(Y) \to H_0(X)$	connected components
$\to$	$In_* : H_1(Y) \to H_1(X)$	closed edge paths
$\to$	$In_* : H_2(Y) \to H_2(X)$	hollow polytopes
	$\vdots$	$\vdots$

If X is a simplicial complex with filtration

$$X_1 \subset X_2 \subset X_3 \subset \dots \subset X_n = X$$

We have inclusion maps

$$X_1 \xrightarrow{} X_2 \longrightarrow \dots \longrightarrow X_i \xrightarrow{} X_{i+1} \longrightarrow \dots \longrightarrow X_{n-1} \xrightarrow{} X_n = X$$

$$\uparrow In_1 \qquad \uparrow In_i \qquad \uparrow In_{n-1}$$

And induced maps for each dimension j

$$H_j(X_1) \to H_j(X_2) \to \dots \to H_j(X_i) \longrightarrow H_j(X_{i+1}) \to \dots \to H_j(X_{n-1}) \to H_j(X)$$

$$\uparrow In_{1*} \qquad \uparrow In_{i*} \qquad \uparrow In_{n-1*}$$

From cycles to bars in a bar code

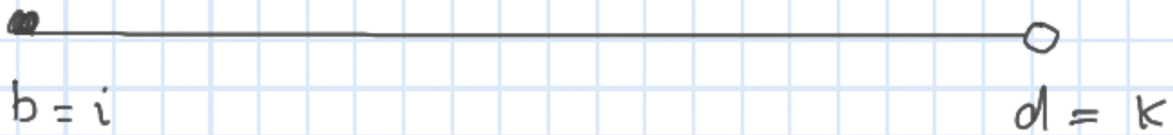
$$\cdots \longrightarrow H_j(X_{i-1}) \longrightarrow H_j(X_i) \longrightarrow \cdots \longrightarrow H_j(X_{k-1}) \longrightarrow H_j(X_k) \longrightarrow \cdots$$

$$\overset{\in}{[z]} \longrightarrow \cdots \longrightarrow \overset{\in}{[z]} \longrightarrow 0$$

$In_{i-1\,*}$ $\qquad$ $In_{k-1\,*}$

$[z]$ is created $\qquad$ $[z]$ is filled in

$\longrightarrow$ $[z] \notin \mathrm{Im}(In_{i-1\,*})$ $\qquad$ $[z] \in \mathrm{Ker}(In_{k-1\,*})$

$[z]$ is born at time i $\qquad$ $[z]$ dies at time k

Persistence Bar