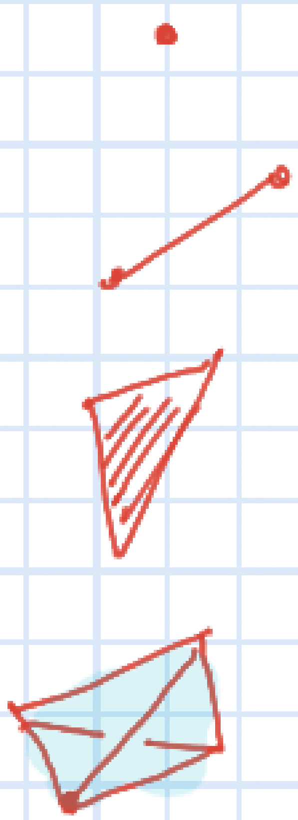
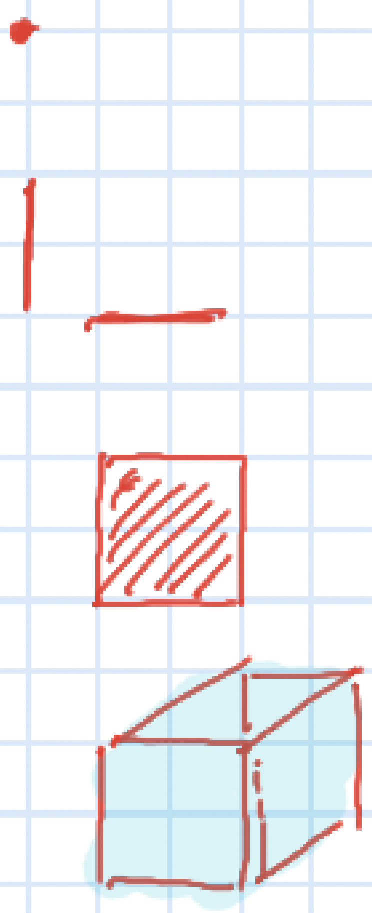
# Cubical Complexes and Cubical Homology

- Useful when analyzing digital image data
- Intuition:



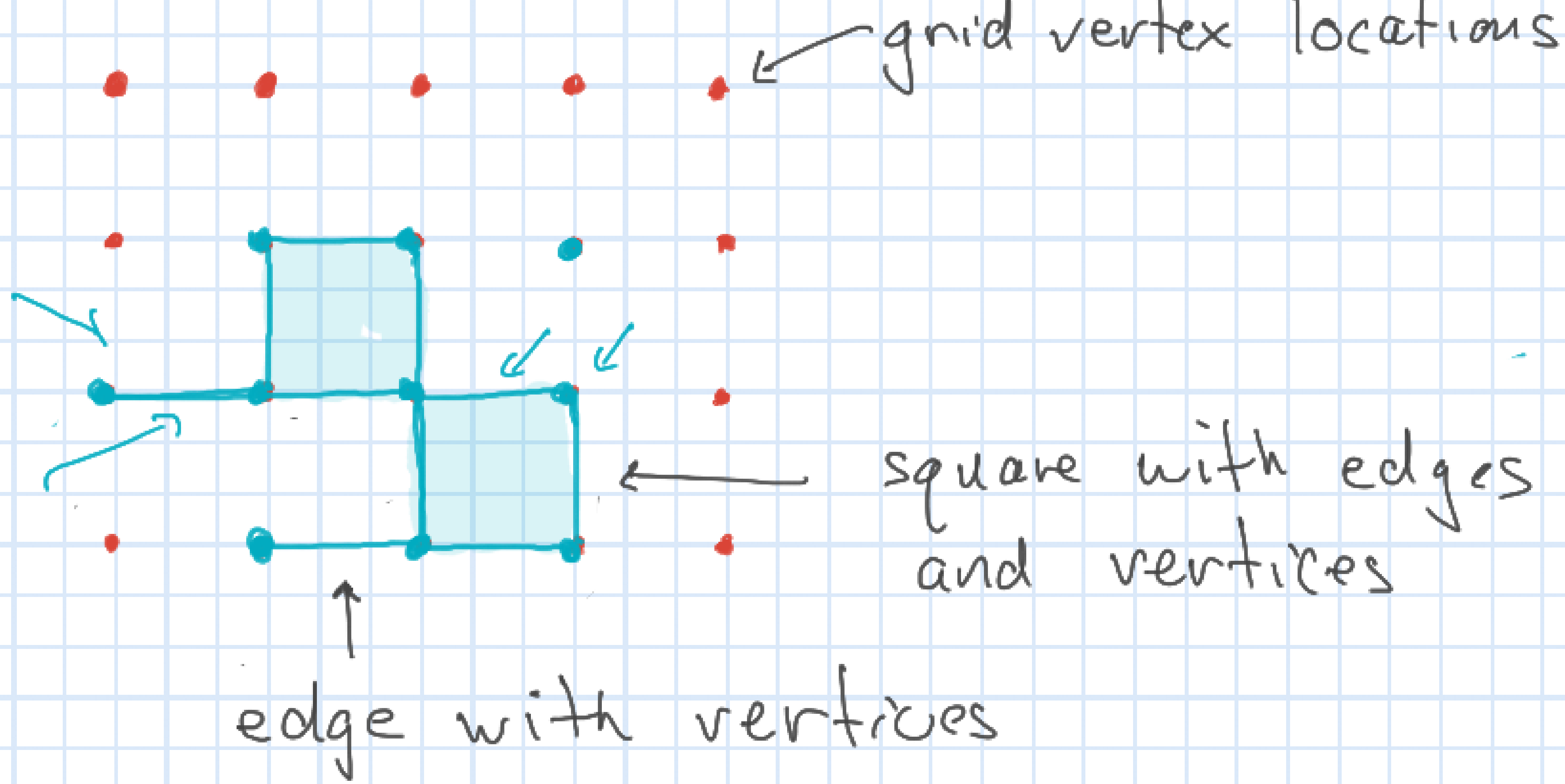
| | Simplicial | | Cubical | |
|---|---|---|---|---|
| Simplices | vertex | $\longleftrightarrow$ | vertex | Cubes |
| | edge | $\longleftrightarrow$ | edge | |
| | triangle | $\longleftrightarrow$ | square | |
| | tetrahedron | $\longleftrightarrow$ | cube | |



- Simplices - located anywhere
  - no alignment
  - any size

Cubes
  - located on a grid
  - must align with grid
  - size fixed by grid

# A small cubical complex

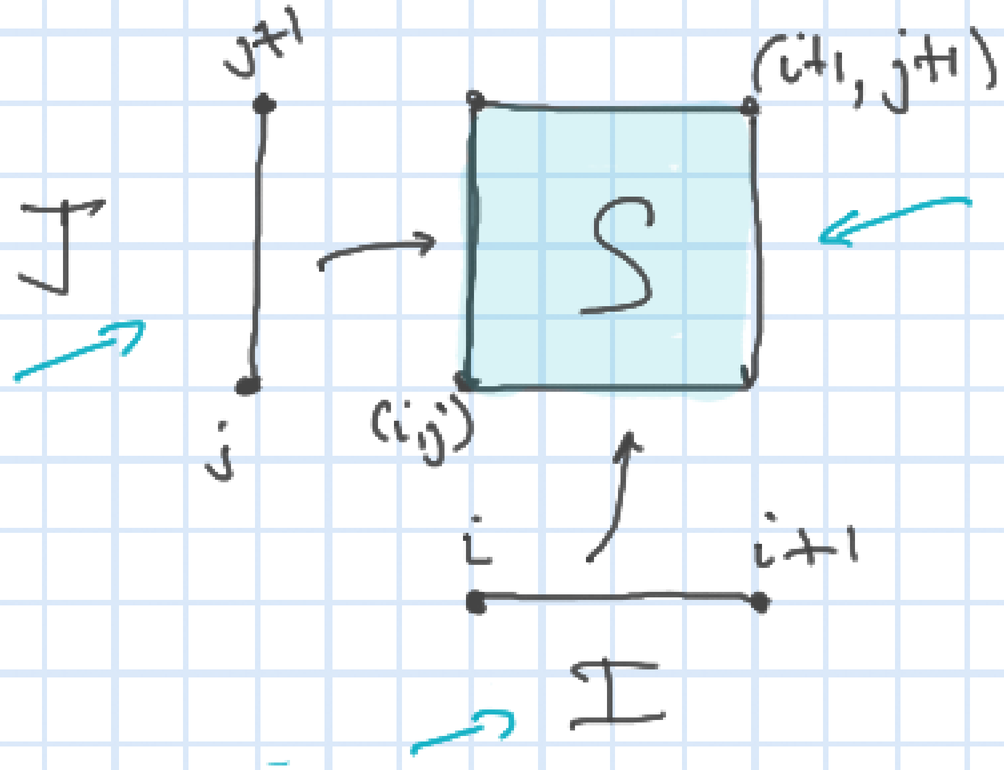



$$X = \{ \text{vertices}, \text{edges}, \text{squares}, \ldots \}$$

$$e \in X \Rightarrow \text{vertices} \in X$$

$$S \in X \Rightarrow \text{edges} + \text{vertices} \in X$$

# Product structure - Geometric




$$S = I \times J$$

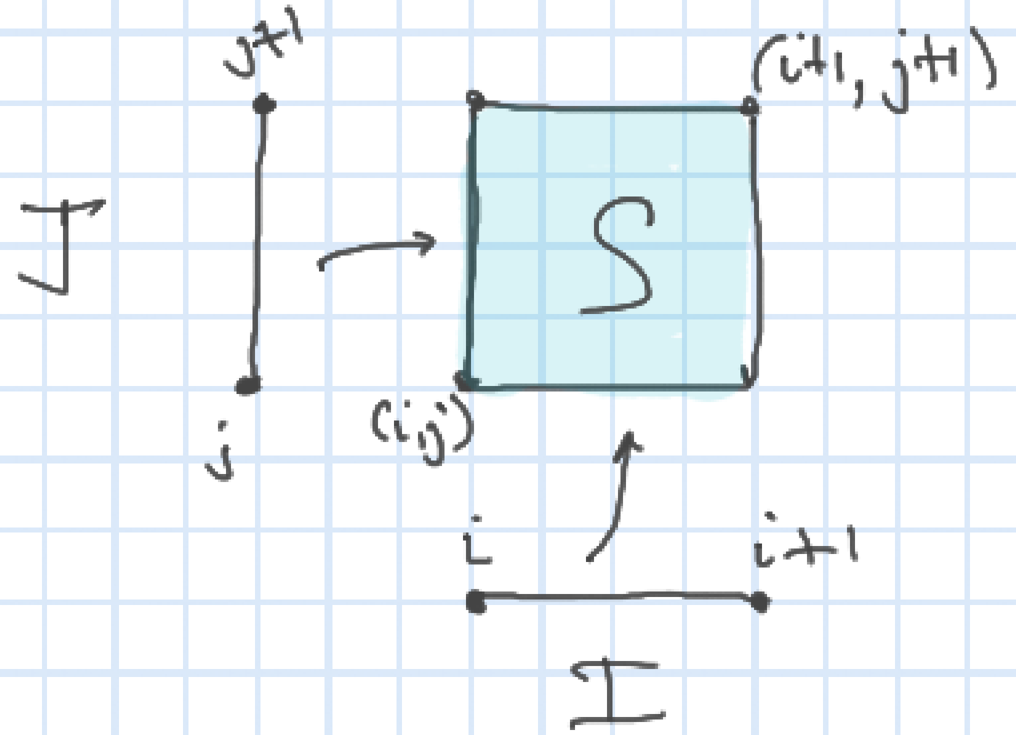
$$= \{(x,y) : i \le x \le i+1, \; j \le y \le j+1\}$$

$$\partial S = I \times \partial J \;\cup\; \partial I \times J$$

$$= \left( [i,i+1] \times [j] \;\cup\; [i,i+1] \times [j+1] \right) \cup \left( [i] \times [j,j+1] \;\cup\; [i+1] \times [j,j+1] \right)$$

} 4 edges

# Product structure - Algebraic




Chain vector spaces/groups generated by elementary cubes: the vertices, edges, ...

Write $\hat{I}$ for the generator of the e.c. $I$

We can define a product on chains

$$\hat{P} \circ \hat{Q} = \widehat{P \times Q} \qquad \text{(algebra)}$$

$P$, $Q$, $P \times Q$, geometric objects

$\hat{P}$, $\hat{Q}$, $\widehat{P \times Q}$ corresponding chains (mod 2)

This can be used to define boundaries

In $\mathbb{R}^1$

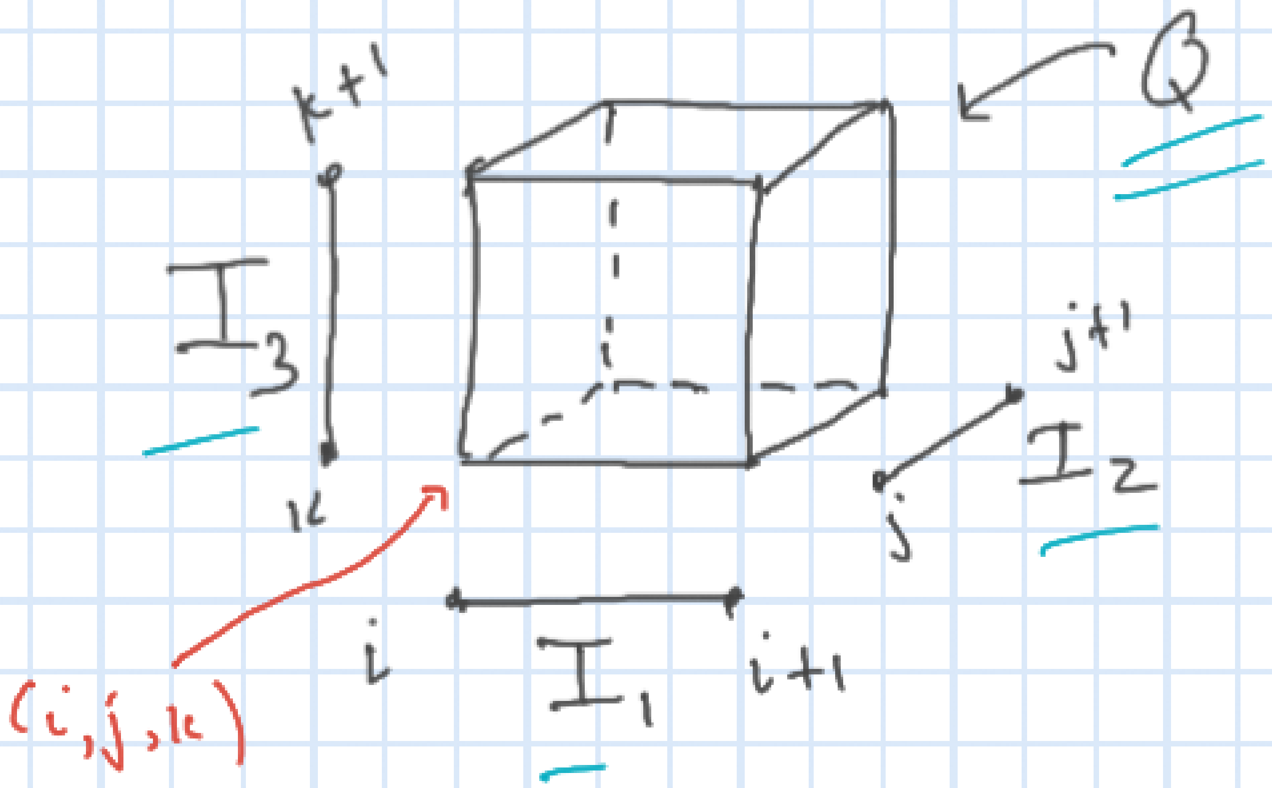
$$\left.\begin{aligned} \partial_0 \, \widehat{[i]} &= 0 \\ \partial_1 \, \widehat{[i, i+1]} &= \widehat{[i+1]} + \widehat{[i]} \end{aligned}\right\} \text{Base case}$$

Define inductively

$$\hat{Q} = \widehat{I_1 \times \dots \times I_k}$$

$$\partial_k \hat{Q} = \partial \hat{I}_1 \diamond \widehat{I_2 \times \dots \times I_k} + \hat{I}_1 \diamond \partial_{k-1} \widehat{I_2 \times \dots \times I_k}$$

Ex.




$$\partial \hat{Q} = \partial \hat{I}_1 \diamond (\hat{I}_2 \diamond \hat{I}_3) + \hat{I}_1 \diamond \partial (\hat{I}_2 \diamond \hat{I}_3)$$

$$\begin{aligned}
&= \widehat{[i+1] \times [j,j+1] \times [k,k+1]} \\
&\quad + \widehat{[i] \times [j,j+1] \times [k,k+1]} && \left.\right\} \begin{matrix} l \\ r \end{matrix} \\
&\quad + \widehat{[i,i+1] \times [j] \times [k,k+1]} \\
&\quad + \widehat{[i,i+1] \times [j+1] \times [k,k+1]} && \left.\right\} \begin{matrix} f \\ \text{back} \end{matrix} \\
&\quad + \widehat{[i,i+1] \times [j,j+1] \times [k]} \\
&\quad + \widehat{[i,i+1] \times [j,j+1] \times [k+1]} && \left.\right\} \begin{matrix} \text{bottom} \\ \text{top} \end{matrix}
\end{aligned}$$

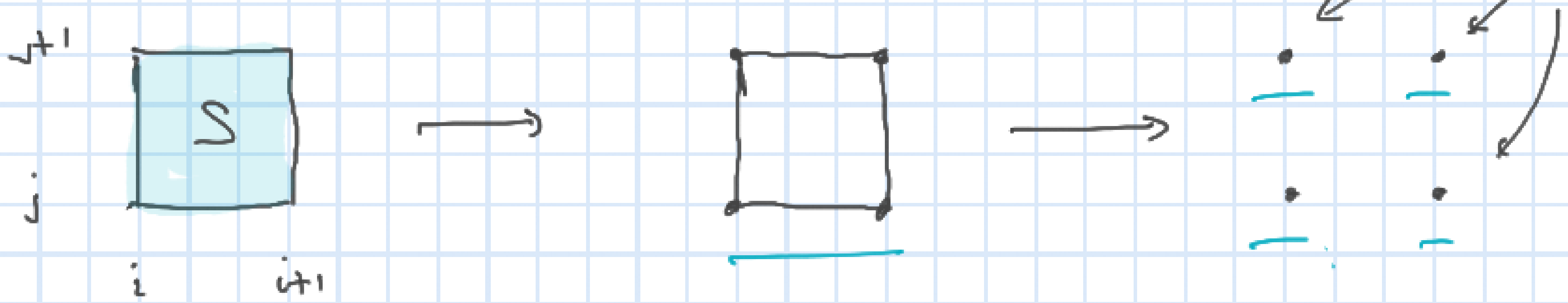
boundary $\Rightarrow$ six faces

As with simplicial homology we define

$$\cdots \xrightarrow{\partial_{i+2}} C_{i+1}(X) \xrightarrow{\partial_{i+1}} C_i(X) \xrightarrow{\partial_i} C_{i-1}(X) \xrightarrow{\partial_{i-1}} \cdots$$

Once again $\partial_i \circ \partial_{i+1} = 0$

2 copies $\equiv 0$ mod 2




$S = [i, i+1] \times [j, j+1]$

$$\partial_1 \circ \partial_2 \left( \widehat{[i,i+1]} \times \widehat{[j,j+1]} \right)$$

$$= \partial_1 \left( \partial_1 \widehat{[i,i+1]} \diamond \widehat{[j,j+1]} + \widehat{[i,i+1]} \diamond \partial_1 \widehat{[j,j+1]} \right)$$

$$= \partial_1 \left( \widehat{[i]} \diamond \widehat{[j,j+1]} + \widehat{[i+1]} \diamond \widehat{[j,j+1]} + \widehat{[i,i+1]} \diamond \widehat{[j]} + \widehat{[i,i+1]} \diamond \widehat{[j+1]} \right)$$

$$= \left\{ \begin{aligned} & \partial_1 \left( \widehat{[i]} \diamond \widehat{[j,j+1]} \right) \\ + & \partial_1 \left( \widehat{[i+1]} \diamond \widehat{[j,j+1]} \right) \\ + & \partial_1 \left( \widehat{[i,i+1]} \diamond \widehat{[j]} \right) \\ + & \partial_1 \left( \widehat{[i,i+1]} \diamond \widehat{[j+1]} \right) \end{aligned} \right. = \left\{ \begin{aligned} & \cancel{\partial_0 \widehat{[i]}}^{\,0} \diamond \widehat{[j,j+1]} + \widehat{[i]} \diamond \widehat{[j]} + \widehat{[i]} \diamond \widehat{[j+1]} \\ + & \cancel{\partial_0 \widehat{[i+1]}} \diamond \widehat{[j,j+1]} + \widehat{[i+1]} \diamond \widehat{[j]} + \widehat{[i+1]} \cdot \widehat{[j+1]} \\ + & \widehat{[i+1]} \diamond \widehat{[j]} + \widehat{[i]} \diamond \widehat{[j]} + \widehat{[i,i+1]} \diamond \cancel{\partial_0 \widehat{[j]}}^{\,0} \\ + & \widehat{[i+1]} \diamond \widehat{[j+1]} + \widehat{[i]} \diamond \widehat{[j+1]} + \widehat{[i,i+1]} \diamond \cancel{\partial_0 \widehat{[j+1]}}^{\,0} \end{aligned} \right.$$

$$= 0$$

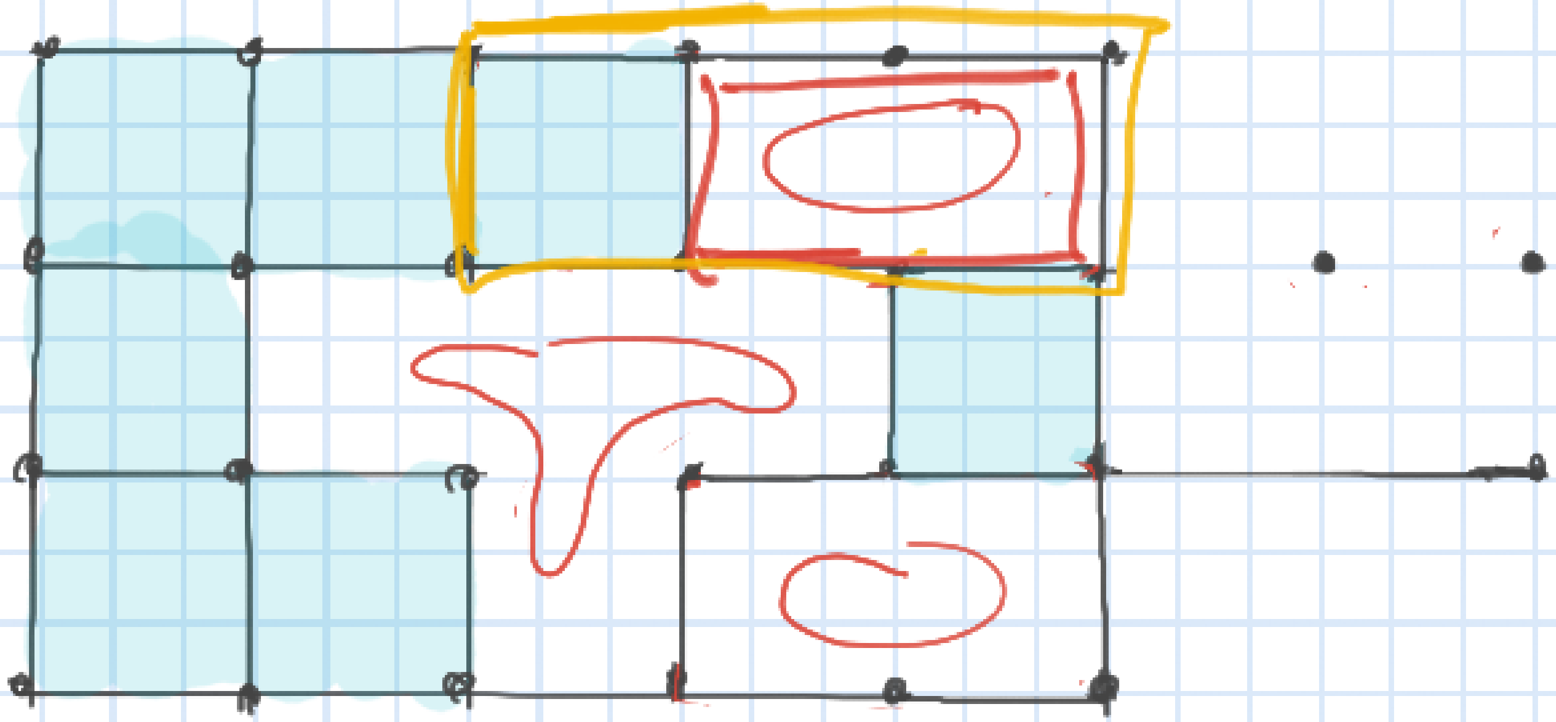
Then

$$Z_i(X) = \operatorname{Ker}(\partial_i)$$
$$B_i(X) = \operatorname{Im}(\partial_{i+1})$$
$$H_i(X) = Z_i(X)/B_i(X)$$

as before.

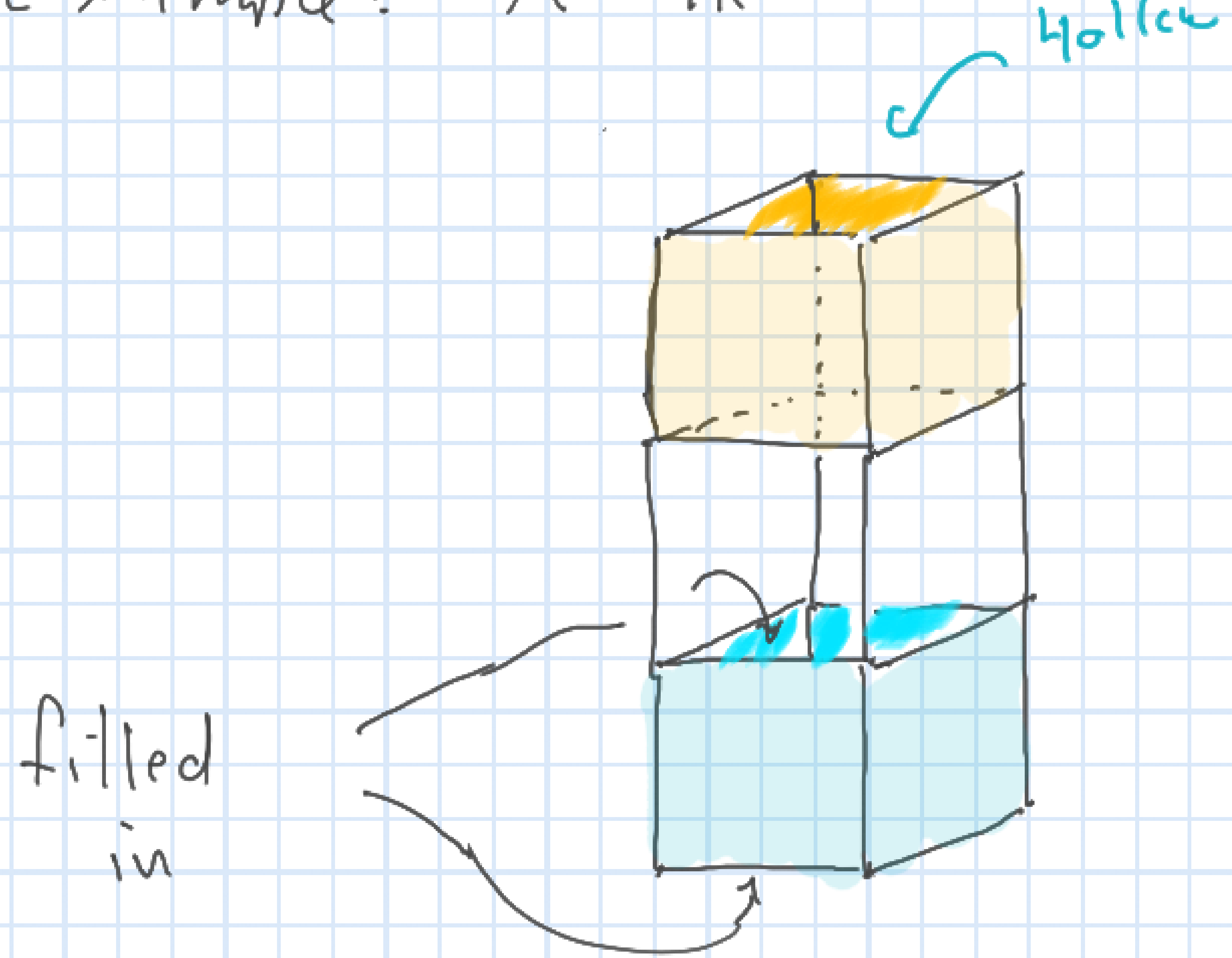
<u>Aside</u>: Axioms for homology

Ex. $X \subset \mathbb{R}^2$



$H_0(X^4) = \mathbb{Z}_2^3$, $\quad H_1(X) = \mathbb{Z}_2^3$

Example: $X \subset \mathbb{R}^3$




$$H_0(X) = \mathbb{Z}_2$$

$$H_1(X) = \mathbb{Z}_2^4$$

$$H_2(X) = \mathbb{Z}_2$$

# Filtrations

- General idea is similar, a nested sequence of cubical complexes

$$X_{a_1} \subset X_{a_2} \subset \dots \subset X_{a_n}$$

- There is no longer a Rips construction

- Important Special Case - Sub-level set Filtrations

Sub-level set filtrations:

Let $X$ be the full complex on the set of points/vertices $V = \{ [i,j] : 1 \le i \le N,\ 1 \le j \le M \}$
( $M$ rows, $N$ columns )

$X$ contains all edges $\overline{(i,j)(i+1,j)}$, $\overline{(i,j)(i,j+1)}$

and squares $[i,i+1] \times [j,j+1]$

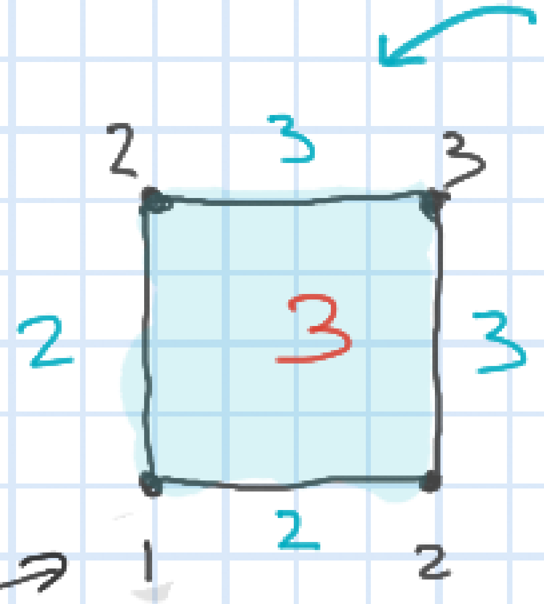
$f : V \longrightarrow A$ , $A$ a set of values.

often $A = \{0, \dots, 255\}$

- Interpret $f$ as a set of color/gray scale/intensity values from an image.

- Extend $f$ to edges and faces

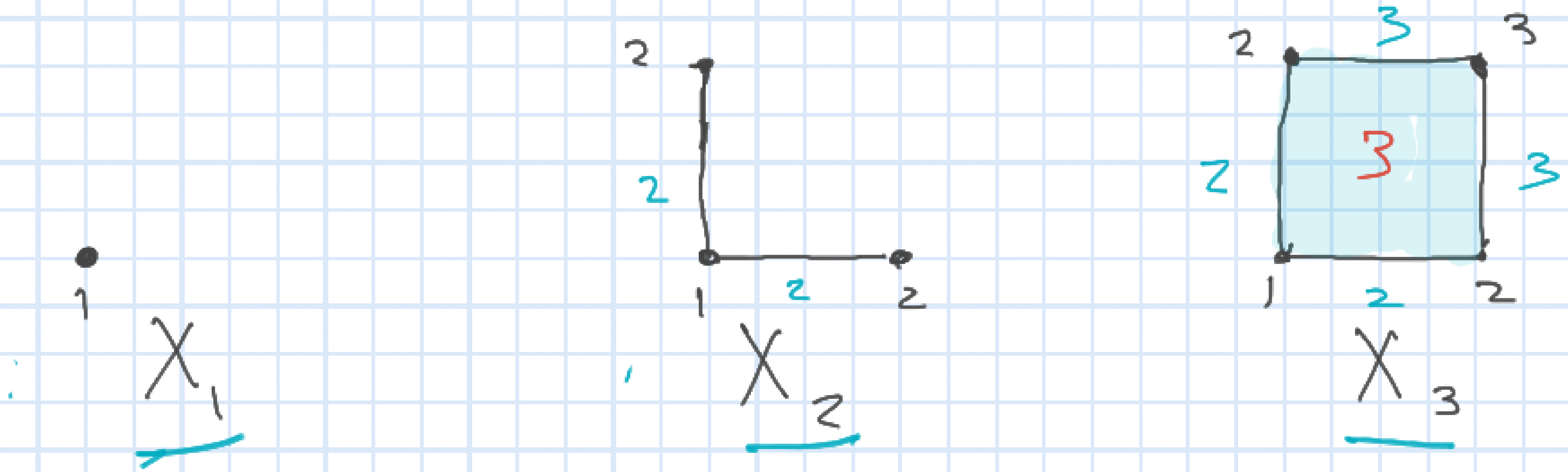
values of $f$




- edge assigned higher of two vertex values

- face assigned highest of edge values

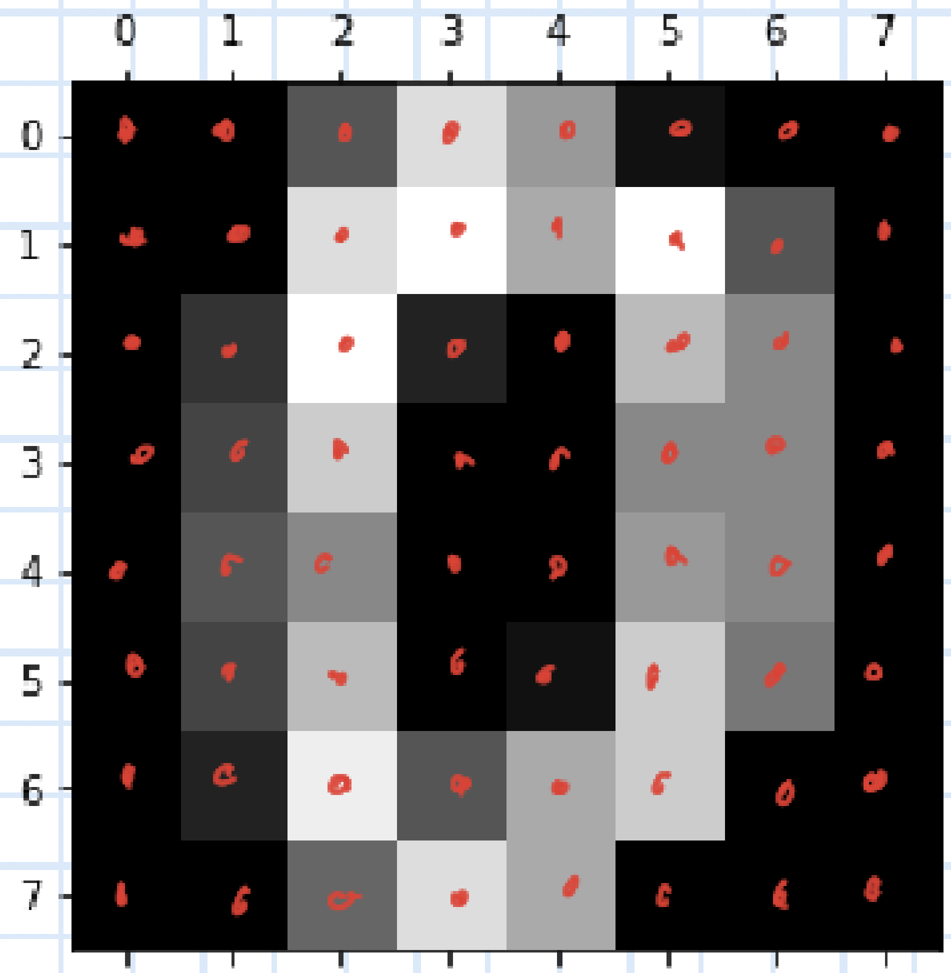
Filtration by increasing sequence of values




Values of edges and faces assigned

$\Rightarrow$ sublevel sets are subcomplexes

$X_{a_i} = \{$ vertices, edges, faces : $f \leq a_i \}$




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15. | 15. | 10. | 2. | 6. | 14. | 15. | 15. |
| 15. | 15. | 2. | 0. | 5. | 0. | 10. | 15. |
| 15. | 12. | 0. | 13. | 15. | 4. | 7. | 15. |
| 15. | 11. | 3. | 15. | 15. | 7. | 7. | 15. |
| 15. | 10. | 7. | 15. | 15. | 6. | 7. | 15. |
| 15. | 11. | 4. | 15. | 14. | 3. | 8. | 15. |
| 15. | 13. | 1. | 10. | 5. | 3. | 15. | 15. |
| 15. | 15. | 9. | 2. | 5. | 15. | 15. | 15. |

Example:

- An image from digits dataset scikit-learn.org, (scikit-learn is a Python machine learning library)
- Values have been reversed from original image

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15. | 15. | 10. | 2. | 6. | 14. | 15. | 15. |
| 15. | 15. | 2. | 0. | 5. | 0. | 10. | 15. |
| 15. | 12. | 0. | 13. | 15. | 4. | 7. | 15. |
| 15. | 11. | 3. | 15. | 15. | 7. | 7. | 15. |
| 15. | 10. | 7. | 15. | 15. | 6. | 7. | 15. |
| 15. | 11. | 4. | 15. | 14. | 3. | 8. | 15. |
| 15. | 13. | 1. | 10. | 5. | 3. | 15. | 15. |
| 15. | 15. | 9. | 2. | 5. | 15. | 15. | 15. |

$\leftarrow X_3$

- A simple filtration: $X_3 \subset X_7 \subset X_{10} \subset X_{15}$

  $X_i$ contains all elementary cubes with value $\leq i$

- Barcodes

  3 7 10 15

  dim 0

  3 7 10 15

  dim 1

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15. | 15. | 10. | 2. | 6. | 14. | 15. | 15. |
| 15. | 15. | 2. | 0. | 5. | 6. | 10. | 15. |
| 15. | 12. | 0. | 13. | 15. | 4. | 7. | 15. |
| 15. | 11. | 3. | 15. | 15. | 7. | 7. | 15. |
| 15. | 10. | 7. | 15. | 15. | 6. | 7. | 15. |
| 15. | 11. | 4. | 15. | 14. | 3. | 8. | 15. |
| 15. | 13. | 1. | 10. | 5. | 3. | 15. | 15. |
| 15. | 15. | 9. | 2. | 5. | 15. | 15. | 15. |

$X_7$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 15. | 15. | 10. | 2. | 6. | 14. | 15. | 15. |
| 15. | 15. | 2. | 0. | 5. | 6. | 10. | 15. |
| 15. | 12. | 0. | 13. | 15. | 4. | 7. | 15. |
| 15. | 11. | 3. | 15. | 15. | 7. | 7. | 15. |
| 15. | 10. | 7. | 15. | 15. | 6. | 7. | 15. |
| 15. | 11. | 4. | 15. | 14. | 3. | 8. | 15. |
| 15. | 13. | 1. | 10. | 5. | 3. | 15. | 15. |
| 15. | 15. | 9. | 2. | 5. | 15. | 15. | 15. |

$X_{10}$

- Barcodes

dim 0


dim 1