The Triangle Inequality for the Bottleneck Distance

Recall:

- If P, Q are persistence diagrams, a partial match M is a bijection between a subset of P and a subset of Q.

- The infinity norm distance between (b,d) and (b',d') is

$$\|(b,d)-(b',d')\|_\infty = \max\{|b-b'|, |d-d'|\}$$

- The infinity norm distance between (b,d) and the diagonal is $\frac{1}{2}(d-b)$

The cost of a partial match M between P and Q is:

$$c(M) = \max \Big\{ \sup_{p_i} \| p_i - M(p_i) \|_\infty ,$$
$$\sup \left\{ \tfrac{1}{2}(d_j - b_j) : p_j \in P \text{ unmatched} \right\},$$
$$\sup \left\{ \tfrac{1}{2}(d_k - b_k) : q_k \in Q \text{ unmatched} \right\} \Big\}$$

Then

$$d_b(P, Q) = \inf_M c(M)$$

taken over all partial matches from P to Q

Suppose P, Q, R are persistance diagrams.

We want to show

$$d_b(P, R) \leq d_b(P, Q) + d_b(Q, R)$$

Since this involves $\| \ \|_\infty$ distances, let's warmup by proving

$$\| \bar{x} - \bar{z} \|_\infty \leq \| \bar{x} - \bar{y} \|_\infty + \| \bar{y} - \bar{z} \|_\infty$$

We write $\bar{x}, \bar{y}, \bar{z}$ since this is a general statement, not relying on persistence points.

$$\|\bar{x}-\bar{z}\|_\infty = \max\{|x_1-z_1|, |x_2-z_2|\}$$

$$\leq \max\{|x_1-y_1|+|y_1-z_1|, |x_2-y_2|+|y_2-z_2|\}$$

by the triangle inequality for $|\ |$.

$$\leq \max\{\max\{|x_1-y_1|, |x_2-y_2|\} + |y_1-z_1|,$$

$$\max\{|x_1-y_1|, |x_2-y_2|\} + |y_2-z_2|\}$$

$$= \max\{|x_1-y_1|, |x_2-y_2|\} + \max\{|y_1-z_1|, |y_2-z_2|\}$$

$$= \|\bar{x}-\bar{y}\|_\infty + \|\bar{y}-z\|_\infty$$

Proving the triangle inequality for $\|\ \|_\infty$

To simplify the proof, adjust the definition of a PD to include all diagonal points. Then the distance to the diagonal $\frac{1}{2}(d_i - b_i)$ of an unpaired point will be the $\|\ \|_\infty$ distance to a point of the other diagram

Since the # of off-diagonal points is still finite, the sup in the definition of cost is still over a finite set. This simplifies the cost

$$c(M) = \sup_{p_i} \| p_i - M(p_i) \|_\infty$$

since there are no unpaired points to consider separately.

Since P, Q, R are finite off the diagonal

There are M_1, M_2, M_3 : $d_b(P, R) = c(M_3)$, $d_b(P, Q) = c(M_1)$, and $d_b(Q, R) = c(M_2)$.

Note: $M_2 \circ M_1$ is a match between P and R

Since M_3 is optimal

$$c(M_3) = \sup_i \| p_i - M_3(p_i) \|$$

$$\leq \sup_i \| p_i - M_2 \circ M_1(p_i) \|_\infty$$

Then by the triangle inequality for $\| \ \|_\infty$

$$\| p_i - M_2 \circ M_1(p_i) \|_\infty \le \| p_i - M_1(p_i) \|_\infty + \| M_1(p_i) - M_2 M_1(p_i) \|_\infty$$

$$\le \sup \| p_j - M_1(p_j) \|_\infty + \sup \| q_k - M_2(q_k) \|_\infty$$

$$= c(M_1) + c(M_2)$$

$$= d_b(P, Q) + d_b(Q, R)$$

So $\quad d_b(P, R) \le d_b(P, Q) + d_b(Q, R)$.

With properties (1) and (2) from the previous session, this pieces d_b is a metric on the set of persistence diagrams