1. What type of function (linear, exponential, power, trig) is the most appropriate for modeling short-term population growth (such as the US population on a scale of a few years, or of a bacterial culture on a scale of ten minutes)? What about modeling long-term population growth? Write a few sentences to justify your answers.

**Solution** In the short term, a linear function is sufficient because the rate of growth is nearly constant over short time intervals. Over the long term, an exponential function is more suitable. Populations grow at a rate proportional to their size, and an exponential function’s derivative is proportional to the function itself. (In the very long term, additional considerations—lack of resources, epidemics, or medical advances, say—can make an exponential model inaccurate.)

2. Let \( f(x) = \frac{\sin x}{x} \) for \( x \neq 0 \).

   (a) What value for \( f(0) \) makes \( f \) continuous at \( x = 0 \)?

   (b) Show that \( \lim_{x \to 0} f'(x) \) exists.

**Solution** By l’Hôpital’s rule, the value \( f(0) = 1 \) makes \( f \) continuous at \( x = 0 \). For part (b), use the quotient rule to see that \( f'(x) = \frac{x \cos x - \sin x}{x^2} \). L’Hôpital’s rule gives

\[
\lim_{x \to 0} \frac{x \cos x - \sin x}{x^2} = \lim_{x \to 0} \frac{-x \sin x + \cos x - \cos x}{2x} = -\lim_{x \to 0} \frac{\sin x}{2} = 0.
\]

(c) Answer the same two questions for \( g(x) = \frac{e^x - 1}{x} \)

**Solution** \( g(0) = 1, \lim_{x \to 0} g'(x) = \frac{1}{2} \).

3. A stone is dropped from a high bridge. After \( t \) seconds, the height of the stone is \( y(t) = 150 - 16t^2 \) feet.

   (a) Without using a graphing calculator, sketch the graph of height as a function of time. What is the domain of the function \( y \)?

   (b) Find the average speed of the stone (including units) over the time interval \( 0.5 \leq t \leq 2 \), and draw the secant line whose slope represents this average speed.
Solution  The stone hits when \( y(t) = 0 \), or \( t = \frac{5}{4} \sqrt{6} \approx 3.062 \) seconds, so the domain is \([0, \frac{5}{4} \sqrt{6}]\). The average speed of the stone on the interval \([0.5, 2]\) is \(-40\) ft/sec.

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4. A savings account is opened on January 1 with $1000 at a constant interest rate of 2% per year, compounded continuously.

(a) Find a formula for the account balance as a function of time, including units.

Solution  The balance is \( B(t) = 1000(1.02)^t \) dollars after \( t \) years.

(b) How many years does it take for the balance to double? To grow to $1,000,000?

Solution  About 35 years; about 349 years.

(c) Find the average rate of earnings for the first quarter of the third year; express your answer in dollars per day.

Solution  The third year starts when \( t = 2 \) (!), so use \( t = 2 \) and \( t = 2.25 \) to compute the change in balance: $5.163. (I do not plan to ask questions whose wording could mislead you in this way.)

(d) Use linear approximation to estimate the amount of interest that accrues on the first day of the tenth year.

Solution  First, \( B'(t) = 1000 \ln(1.02)(1.02)^t \) dollars per year after \( t \) years, or about 0.054254(1.02)\(^t\) dollars per day after \( t \) years. The interest accrued on the first day of the tenth year is roughly 0.054254(1.02)^{9} \approx 0.065
5. Let \( f(x) = (x^2 - 1)^2 \). Find the critical point(s) and inflection point(s) of \( f \) by making sign diagrams for the first and second derivatives. Classify the critical point(s) as local minima, local maxima, or neither. Use the information you find to make a careful sketch of the graph of \( f \) on the interval \([-2, 2]\). Do not use a graphing calculator.

**Solution**  
Done in class. Critical points are \( x = \pm 1 \) (local minima) and \( x = 0 \) (local maximum), the inflection points are \( x = \pm 1/\sqrt{3} \). Calculate the function values at these points, then interpolate, using your knowledge of the concavity of the graph.

6. The period of a satellite’s orbit is proportional to the 3/2 power of the radius of the orbit, measured from the center of the earth.

(a) If an orbit’s radius is increased by 10\%, by about how much does the period change? Use linear approximation, and show all work.

**Solution**  
Let \( T \) be the period of the orbit in minutes, \( r \) the radius of the orbit in miles. (You should decide on the names of variables, including units, as the first step of every word problem you do!) We are given that \( T = kr^{3/2} \) for some positive constant \( k \).

Linear approximation says that a small change \( \Delta r \) in the radius gives rise to a small change \( \Delta T \approx \frac{dT}{dr}\Delta r = \frac{3}{2}kr^{-1/2}\Delta r \) in the period. Dividing this equation by the original relation between \( r \) and \( T \) gives the fractional change in \( T \) given a fractional change in \( r \):

\[
\frac{\Delta T}{T} = \frac{3}{2} \frac{\Delta r}{r}.
\]

If the radius is increased by 10\% \((\Delta r/r = 0.1)\), the period increases by about 15\%.

(b) The space shuttle orbits at an altitude of 150 miles, and takes 90 minutes to complete one orbit. Estimate the new orbital period if the altitude is increased to 200 miles. The radius of the earth is 3900 miles.

**Solution**  
The initial radius is 3900 + 150 = 4050 miles, and the change in the orbital radius is 200 – 150 = 50 miles, so the fractional change in the radius is \( \Delta r/r = 50/4050 \approx 0.0123 \), or about 1.23\%. The change in orbital period is about 90 \cdot \frac{3}{2}(\Delta r/r) = 1\frac{2}{3} \) minutes, so the new period is 91\frac{2}{3} minutes.
7. Water drains at 3 cubic feet per minute from a rain collector whose shape is an inverted cone of radius 6 feet and height 4 feet. Assuming the tank starts full, how rapidly is the water level falling initially? How rapidly is the level falling when the depth of the water is 1 foot? Express your answers in feet per minute, feet per second, and inches per second. The volume of a cone of radius $r$ and height $h$ is $V = \frac{1}{3} \pi r^2 h$.

**Solution**  Let $h$ be the depth of water in feet, and let $r$ be the radius of the surface of the water at depth $h$. For this cone, $r = 1.5h$, so when the depth is $h$, the volume in the tank is

$$V = \frac{1}{3} \pi r^2 h = \frac{3}{4} \pi h^3,$$

so

$$\frac{dV}{dt} = \frac{9}{4} \pi h^2 \frac{dh}{dt}.$$

We are given that $dV/dt = -3$, so when the tank is full ($h = 4$), we have $dh/dt = -4/(3\pi h^2) = -0.0265$ ft/min. When the depth is 1, the rate of decrease is $dh/dt = -0.4244$ ft/min. Divide by 60 to get ft/sec, and divide by 60 = 5 to get in/sec.

8. The cross-sectional area of a tree increases by about the same amount each year, since the supply of nutrients is roughly constant, and the cross-sectional area is proportional to the volume of new tissue added.

(a) Do the annual growth rings get thicker or thinner as the tree ages?

**Solution**  Thinner. (This is an average, idealized statement. Differing amounts of rain and sun play an important role in the amount of annual growth.)

(b) A five-year-old tree has radius 2 inches. Approximately how thick will the next ring be? That is, what will be the radius of the tree when it is 6 years old? (Suggestion: Use related rates.)

**Solution**  Let $t$ be the age of the tree in years, $r$ the radius in inches. We are given that $r^2 = kt$ for some positive constant $k$, so $2r(dr/dt) = k$. (Now we see quantitatively why the rings get thinner as $r$ gets larger.) Using $k = r^2/t$ and solving for $dr/dt$, we find that $dr/dt = r/(2t)$. If $r = 2$ and $t = 5$, we expect the next ring (whose thickness is roughly $dr/dt$) to be about $1/5 = 0.2$ inches thick. (It’s not a bad exercise to estimate the amount of growth algebraically; the result is about 0.19 in.)
(c) About how old will the tree be when its radius is 10 inches?

Solution Since \( r = 2 \) when \( t = 5 \), we find that \( k = 4/5 = 0.8 \), so \( r^2 = 0.8t \). Setting \( r = 10 \) and solving for \( t \), we find that the tree has radius 10 inches when it is 125 years old.

9. A car drives along a straight highway. The horizontal position of a point on the tread of a tire is given by \( x(t) = 80t - 1.25 \sin(64t) \) feet after \( t \) seconds. Find the maximum forward speed of a point on the tread. At what point of the rotation does this maximum forward speed occur? Show that the point comes to a stop once per revolution.

Solution The forward speed is \( x'(t) = 80 - 80 \cos(64t) \), which is at most 160 ft/sec (when the point is above the axle). Once per revolution we have \( \cos(64t) = 1 \) (when the point is directly below the axle, i.e., touching the ground), and at these times \( x'(t) = 0 \).

10. A rumor spreads in a population of \( M \) people. After \( t \) days, \( P \) people have heard the rumor. To a good approximation, 

\[
\frac{dP}{dt} = kP(M - P)
\]

for a positive constant \( k \) that measures how frequently the rumor is repeated.

(a) Using the fact that \( dP/dt \) is the number of people per day who hear the rumor for the first time, explain in words why the equation above holds.

Solution At time \( t \), \( P \) people have heard the rumor and \( M - P \) people have not. The rate of spread is equal to the number of incidents (per unit time) in which one of the \( P \) people who have heard tells the rumor to one of the \( M - P \) people who have not heard. The number of such meetings is jointly proportional to \( P \) and \( M - P \).
(b) How many people have heard the rumor when the rate of spread is largest?

**Solution** Think of the differential equation as saying \( dP/dt = f(P) \), with \( f(P) = kP(M - P) = k(MP - P^2) \). The graph of \( f \) is a parabola that opens downward and that crosses the horizontal axis at \( P = 0 \) and \( P = M \). The maximum of \( f \) therefore occurs halfway between, at \( P = M/2 \). Since the rate of spread is \( f(P) \), the rate of spread is most rapid when half the people have heard (and half have not heard).

(c) Compute \( d^2P/dt^2 \); show that the graph of \( P \) as a function of \( t \) is concave up when \( P < M/2 \), and is concave down when \( P > M/2 \). Does this confirm your answer to part (b)? Explain.

**Solution** The chain rule gives
\[
\frac{d^2P}{dt^2} = k(M - 2P) \frac{dP}{dt} = k^2(M - 2P)P(M - P).
\]
A sign diagram for the right hand-side reveals that \( d^2P/dt^2 > 0 \) if \( 0 < P < M/2 \), while \( d^2P/dt^2 < 0 \) if \( M/2 < P < M \). Again we see that the rate of spread increases to a maximum (\( dP/dt \) is increasing until \( P = M/2 \)) and then decreases, so the maximum rate of spread happens when \( P = M/2 \).

11. A spherical balloon is inflated at a constant rate of 25 in\(^3\)/sec.

(a) When the radius is 4 inches, how rapidly is the radius increasing?

(b) Assuming the balloon never pops, what is the limit of \( dr/dt \) as \( t \to \infty \)?

**Solution** Letting \( r \) be the radius in inches and \( V \) be the volume in cubic inches, we have
\[
V = \frac{4}{3} \pi r^3, \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.
\]
Since we are given \( dV/dt = 25 \), we find that \( dr/dt = 25/(4\pi r^2) \) in/sec. When \( r = 4 \), \( dr/dt \approx 0.124 \) in/sec. As \( r \to \infty \), \( dr/dt \to 0 \).

12. A cylindrical can of radius \( r \) and height \( h \) must hold 1 liter (1000 cc). Find the dimensions that minimize the surface area \( A = 2\pi r^2 + 2\pi rh \). The volume of a cylinder is \( V = \pi r^2h \), and your answer should be given in both symbolic and numerical form.
**Solution**  We are given that $1000 = \pi r^2 h$, or $r h = 1000/r$. As a function of $r$, the surface area is $A(r) = 2\pi r^2 + 2000/r$. Taking the derivative, $A'(r) = 4\pi r - 2000/r^2$, which is zero exactly when $r^3 = 2000/(4\pi)$, or $r = \sqrt[3]{500/\pi} \approx 5.42$ in. The height would be $h = 1000/(\pi r^2) \approx 10.83$ in.

13. Consider the curve $x^3 + y^3 - 6xy = 0$. Compute the slope of the tangent line at $(x,y)$. Find the equations of the tangent lines at the points $(3,3)$ and $(4/3,8/3)$.

**Solution**  Implicit differentiation gives $3x^2 + 3y^2 y'(x) - 6y - 6xy'(x) = 0$, or

$y'(x) = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$.

At $(3,3)$, the slope is $-1$, while at $(4/3,8/3)$ the slope is $4/5$. The tangent lines are $y = 6 - x$ or $x + y = 6$, and $y - 8/3 = (4/5)(x - 4/3)$.

14. Find the tangent line to the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$ at the point $(x_0,y_0)$. Show that the sum of the $x$- and the $y$-intercepts is $c$, regardless of the point of tangency.

**Solution**  Implicit differentiation gives $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2} \frac{dy}{dx} = 0$, so the slope of the tangent line at $(x_0,y_0)$ is $m = -(y_0/x_0)^{1/2}$ and the equation is $y - y_0 = m(x - x_0)$ for this value of $m$. We want to find the sum of the $x$- and $y$-intercepts. Each intercept is found by setting a variable to 0 and solving for the other. Thus, the $y$-intercept is $y_0 - mx_0 = y_0 + \sqrt{x_0}y_0$ and the $x$-intercept is $x_0 - y_0/m = x_0 + \sqrt{x_0}y_0$. The sum of the intercepts is

$$(x_0 + \sqrt{x_0}y_0) + (y_0 + \sqrt{x_0}y_0) = x_0 + 2\sqrt{x_0}y_0 + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c,$$

regardless of the point of tangency.

15. Use calculus to show that the function $f(x) = \arctan(x) + \arctan(1/x)$ is constant. Explain this fact geometrically. What is the constant?

**Solution**  By the chain rule,

$$f'(x) = \frac{1}{1 + x^2} + \frac{-1/x^2}{1 + 1/x^2} = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$$

for every $x \neq 0$. A right triangle of base 1 and height $x$ shows that the constant value of $f$ is $\pi/2$: We have $\tan \theta_1 = x$ (or $\theta_1 = \arctan x$) and $\tan \theta_2 = \frac{1}{x}$ (or $\theta_2 = \arctan \frac{1}{x}$), see figure on next page.
16. Two cars start moving from the same point; the first travels south at 60 mph and the second travels west at 25 mph. At what rate is the distance between the cars increasing after two hours?

**Solution** Let $t$ be time measured in hours, and consider coordinates with the origin at the cars’ starting point, with the positive $y$-axis pointing south (upside-down from normal maps) and with distances in miles. The cars’ respective positions are $c_1(t) = (0, 60t)$ and $c_2(t) = (25t, 0)$. By the Pythagorean theorem, the distance (in miles) between the cars after $t$ hours is

$$L(t) = \sqrt{c_1^2(t) + c_2^2(t)} = t\sqrt{60^2 + 25^2} = 65t.$$ 

The distance between the cars increases at a steady speed of 65 mph, regardless of how much time has passed. (For extra practice, answer the same question assuming the second car departs one hour after the first car. In this event, the elapsed time does matter.)

17. A street light is at the top of a 15 foot post. A person 6 feet tall walks along a straight sidewalk underneath at 5 feet/sec. When they are 40 feet from the post, how rapidly is the length of their shadow increasing?

**Solution** Assume the person walks beneath the light at $t = 0$, with time measured in seconds. Let $x$ be the distance (in feet) to the base of the light pole, and let $\ell$ be the length of their shadow (measured from the base of the pole). Use similar triangles to find the length of the person’s shadow, see left-hand figure on next page, then use related rates to find $d\ell/dt$ when $x = 40$.

18. The top of a 10-foot ladder rests against a vertical wall, and the base slides away from the wall at 2 feet/sec.

(a) When the base of the ladder is 6 feet from the wall, how rapidly is the top of the ladder dropping?

(b) When the angle between the ladder and the wall is 45°, how rapidly is the angle changing? (Be careful with units!)
**Solution**  With time $t$ measured in seconds, let $y(t)$ be the height of the top of the ladder at time $t$, $x(t)$ the distance from the wall to the base of the ladder, and $\theta(t)$ the angle made by the wall and the top of the ladder, see right-hand figure below. The related rates calculations are left to you. (Note carefully that angles are measured in radians, so the trig functions have the expected derivatives; you must convert your answer to degrees per second.)

![Diagram of a ladder against a wall with labels for $y$, $x$, $\theta$, and measurements of 15, 6, and 10.]

19. A car traveling 60 mph brakes to a stop in 5 seconds. If we have measurements of the speed as a function of time, we can use left- and right-hand sums to estimate the stopping distance.

(a) Are the left-hand sums under- or over-estimates? Explain.

**Solution**  Over-estimates: The speed at the left-hand endpoint is larger than the speed at subsequent times.

(b) Assuming the speed measurements are taken at equal intervals, how many measurements would be required to estimate the stopping distance to within one foot, that is, to make the difference between upper and lower estimates smaller than 2 feet?

**Solution**  The change in speed is 88 ft/sec, over a time interval of length 5 sec. If $n$ intervals are used (that is, $(n + 1)$ measurements), the difference between left- and right-hand sums is $|f(b) - f(a)| \cdot (b - a)/n = 440/n$ ft. To make this smaller than 2 ft, we need at least 220 intervals, or 221 measurements.
20. If we want to estimate $\int_0^1 e^{-x^2} \, dx$ to an accuracy of 0.01 using left- and right-hand sums with equally-spaced points, what is the smallest number of intervals we need?

**Solution**  We want $n$ such that $|e^0 - e^{-1}|/n$ is at most 0.02, or $(1 - e^{-1})/n \leq 0.02$, so $n \geq 32$.

21. Without evaluating the integral, determine whether $\int_0^\pi e^x \cos x \, dx$ is positive. (Hint: The interval contains two half-arches of cos. If you can, express the given integral in terms of $\int_0^{\pi/2} e^x \cos x \, dx$.)

**Solution**  The integral is negative. In fact, $\int_0^\pi e^x \cos x \, dx = (1-e^{\pi/2}) \int_0^{\pi/2} e^x \cos x \, dx$.

22. Let $f'$ be the function below. On the grid provided, carefully sketch the function $f$, assuming $f(0) = 0$. Hint: The graph of $f$ has no corners.