4.7 Theorems about Continuous & Differentiable Functions

Recall

p.46 A continuous function has a graph that can be drawn without lifting the pencil from the paper.

p.79 A function $f$ is differentiable for any $x$-value where $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ exists.

p.97 If $f(x)$ is differentiable at a point $x=a$, then $f(a)$ is continuous at $x=a$.

New

p.107 If $f$ continuous on $[a,b]$ then $f$ has a global max & a global min on $[a,b]$.

p.162 Suppose $f$ is defined on an interval $[a,b]$ and has a local max or min at $x=c$ which is not an endpoint. If $f$ is differentiable at $x=a$, then $f'(a)=0$. 

**The Mean-Value Thm**

If $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$, then there exists a number $c$, with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$ 

(See discussion & picture on p.208)

**The Increasing Fcn Thm**

Suppose that $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$.

1. If $f'(x) > 0$ on $(a,b)$, then $f$ is increasing on $[a,b]$.

2. If $f'(x) \geq 0$ on $(a,b)$, then $f$ is nondecreasing on $[a,b]$.

**The Constant Fcn Thm**

Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$. If $f'(x) = 0$ on $(a,b)$ then $f$ is constant on $[a,b]$. 
Theorem 6.284

The Race Track Principle

Suppose \( g \) & \( h \) are continuous on \([a,b]\) & differentiable on \((a,b)\) & suppose that \( g'(x) \leq h'(x) \) for \( a < x < b \).

- If \( g(a) = h(a) \) then \( g(x) \leq h(x) \) for \( a \leq x \leq b \).
- If \( g(b) = h(b) \) then \( g(x) \geq h(x) \) for \( a \leq x \leq b \).

Interpretation

Think of \( g(x) \) & \( h(x) \) as positions of 2 horses on a racetrack.

Horse \( h(x) \) always moves faster than \( g(x) \).

If they start together, horse \( h(x) \) is always ahead.

If they finish together, horse \( g(x) \) was ahead during whole race.
Example

Use Racetrack Principle to show that \( e^x > 1+x \) for all \( x \).

Proof

Call \( g(x) = 1+x \).

Call \( h(x) = e^x \).

Note \( g'(x) = 1 \) and \( h'(x) = e^x \).

We know from graph that \( h'(x) \geq g'(x) \) for all \( x \geq 0 \).

\[ \begin{align*}
\text{Observe that } g(0) = h(0) = 1 \text{ so they start together} & \quad \text{for all } 0 \leq x < \infty, \text{ use } a = 0 \\
\text{know that } h(x) \geq g(x). & \end{align*} \]

Now, when \( x < 0 \), we can see on graph that \( h'(x) \leq g'(x) \) ... things have flipped. ... Going from \(-\infty < x \leq 0\)

Think of \( b = 0 \) and so \( g(x) \leq h(x) \) for all \( x \leq 0 \).

Thus, \( 1+x \leq e^x \forall x \)