College of the Holy Cross, Spring 2009
Math 132, Midterm Exam 1
Thursday, February 19
Solutions

Formulas that may be useful:

$$\cos^2(t) = \frac{1+\cos(2t)}{2}, \quad \sin^2(t) = \frac{1-\cos(2t)}{2}$$

1. $\displaystyle\int \frac{1}{a^2+u^2}\, du = \frac{1}{a}\arctan\frac{u}{a} + C$

2. $\displaystyle\int \sqrt{u^2-a^2}\, du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

3. $\displaystyle\int \sqrt{u^2+a^2}\, du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2+a^2}| + C$

4. $\displaystyle\int \sqrt{a^2-u^2}\, du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C$

5. $\displaystyle\int u^2\sqrt{a^2-u^2}\, du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\frac{u}{a} + C$

6. $\displaystyle\int \sqrt{2au-u^2}\, du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arccos\left(\frac{a-u}{a}\right) + C$

7. $\displaystyle\int \sec(\theta)\, d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$

8. $\displaystyle\int \csc(\theta)\, d\theta = \ln|\csc(\theta)-\cot(\theta)| + C$

I.

A. (5) State the definition of the integral $\int_a^b f(x)\,dx$. Explain all terms involved.

Solution: If $f(x)$ is continuous on $[a,b]$ and $\Delta x = \frac{b-a}{n}$, then the definite integral is the limit of the Riemann sums

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

for all choices of $x_i^* \in [x_{i-1}, x_i]$.

B. (5) State the Fundamental Theorem of Calculus (not the evaluation theorem).

Solution: If $f(x)$ is continuous on $[a,b]$, and $F(x)$ is defined as $F(x) = \int_a^x f(t)\,dt$ for all $a < x < b$, then F is an antiderivative of f, *i.e.*, $F'(x) = f(x)$.

II. Compute the derivatives of each of the following functions defined by integrals.

A. (5) $f(x) = \int_{\sqrt{2}}^{x} \ln(t^2)\,dt$

Solution: By the Fundamental Theorem of Calculus, $f'(x) = \ln x^2$.

B. (5) $g(x) = \int_{x^3}^{1} e^{t^2}\,dt$

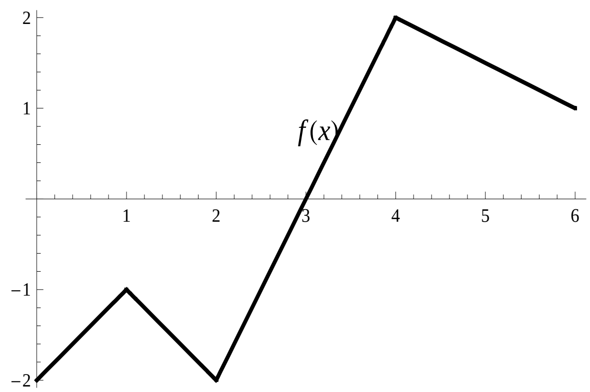
Solution: $g(x) = \int_{x^3}^{1} e^{t^2}\,dt = -\int_{1}^{x^3} e^{t^2}\,dt = -h(x^3)$, where $h(x) = \int_1^x e^{t^2}\,dt$.

By the Chain Rule, $g'(x) = -h'(x^3)\cdot 3x^2$.

By the Fundamental Theorem of Calculus. $h'(x) = e^{t^2}$.

Thus, $g'(x) = -e^{(x^3)^2}\cdot 3x^2 = -e^{x^6}\cdot 3x^2$.

III. The following graph (made up of straight line segments) shows $y = f(t)$ for $0 \le t \le 6$.



The function F is defined by $F(x) = \int_0^x f(t)\ dt$.

A. (5) Determine the values $F(x)$ for $x = 0, 1, 2, 3, 4, 6$ and enter them in the following table.

Solution: We use the interpretation of the definite integral as signed area.

x	0	1	2	3	4	6
$F(x)$	0	$-\frac{3}{2}$	-3	-4	-3	0

B. (5) Does $F(x)$ have any critical points? If so, say where. If not say why not.

Solution: Since $F'(x) = f(x)$ and $f(3) = 0$, F has a critical point at $x = 3$.

C. (5) Over which interval(s) is $F(x)$ concave up?

Solution: F is concave up where F' is increasing. Thus, F is concave up on $(0, 1) \cup (2, 4)$.

IV.

A. (5) Integrate with a suitable u-substitution: $\displaystyle\int \frac{\cos(\pi\sqrt{x})}{\sqrt{x}}\ dx$.

Solution: We make the substitution $u = \pi\sqrt{x}$, $du = \dfrac{\pi}{2\sqrt{x}}\,dx$. Then, the given integral equals $\displaystyle\frac{2}{\pi}\int \cos u\ du = \frac{2}{\pi}\sin u + C = \frac{2}{\pi}\sin(\pi\sqrt{x}) + C$.

B. (5) Integrate with a suitable u-substitution: $\int \sin^2 x \cos^3 x \ dx$.

Solution: Since one of the trigonometric functions appears to an odd power, we rewrite the integral as $\int \sin^2 x \cos^2 x \cos x \ dx = \int \sin^2 x(1-\sin^2 x) \cos x \ dx$ and make the substitution $u = \sin x$, $du = \cos x \, dx$. Then, the integral equals $\int u^2(1-u^2) \ du = \int (u^2 - u^4) \ du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$

C. (5) Integrate by parts: $\int_1^e x^5 \ln x \ dx$

Solution: We take $u = \ln x$, $u' = 1/x$, $v' = x^5$, $v = x^6/6$. Then

$$\int_1^e x^5 \ln x \ dx = \frac{x^6}{6} \ln x \Big|_1^e - \int_1^e \frac{1}{x}\frac{x^6}{6} \ dx = \frac{e^6}{6} - \frac{1}{6}\int_1^e x^5 \ dx = \frac{e^6}{6} - \frac{1}{6}\frac{x^6}{6}\Big|_1^e = \frac{e^6}{6} - \frac{e^6}{36} + \frac{1}{36}.$$

D. (10) Integrate with the partial fraction method: $\int_{-2}^{-1} \frac{x^3+10x^2-8x+57}{x^2+10x-11} \ dx$

Solution: First we perform long division to obtain $\frac{x^3+10x^2-8x+57}{x^2+10x-11} = x + \frac{3x+57}{x^2+10x-11}$. The denominator factors as $(x-1)(x+11)$. We have

$$\int_{-2}^{-1} \frac{x^3+10x^2-8x+57}{x^2+10x-11} \ dx = \int_{-2}^{-1} \left(x + \frac{3x+57}{x^2+10x-11} \right) \ dx = \frac{x^2}{2}\Big|_{-2}^{-1} + \int_{-2}^{-1} \frac{3x+57}{x^2+10x-11} \ dx.$$

For the last integral, we use partial fractions.

$\frac{3x+57}{x^2+10x-11} = \frac{A}{x-1} + \frac{B}{x+11}$. Thus $A(x+11) + B(x-1) = 3x + 57$.

If $x = 1$, we have $12A = 60$ and thus, $A = 5$.

If $x = -11$, we have $-12B = 24$ and thus, $B = -2$.

Therefore, $\int_{-2}^{-1} \frac{x^3+10x^2-8x+57}{x^2+10x-11} \ dx = \frac{x^2}{2}\Big|_{-2}^{-1} + 5\int_{-2}^{-1} \frac{1}{x-1} \ dx - 2\int_{-2}^{-1} \frac{1}{x+11} \ dx = \left(\frac{x^2}{2} + 5\ln|x-1| - 2\ln|x+11| \right)_{-2}^{-1} = -\frac{3}{2} + 5\ln 2 - 5\ln 3 - 2\ln 10 + 2\ln 9.$

E. (10) Integrate via trigonometric substitution: $\int \frac{1}{x\sqrt{x^2+9}}\,dx$

Once you get to an integral of trigonometric functions, you can use the table to find the integral.

Solution: We make the substitution $x = 3\tan t,\ dx = 3\sec^2 t\,dt$. Then $\int \frac{1}{x\sqrt{x^2+9}}\,dx =$ $\int \frac{1}{3\tan t\sqrt{9\tan^2 t+9}} 3\sec^2 t\,d = \frac{1}{3}\int \frac{1}{\tan t\sqrt{\tan^2+1}}\sec^2 t\,dt = \frac{1}{3}\int \frac{1}{\tan t\sqrt{\sec^2 t}}\sec^2 t\,dt =$ $\frac{1}{3}\int \frac{1}{\tan t\sec t}\sec^2 t\ dt = \frac{1}{3}\int \frac{1}{\tan t}\sec t\ dt = \frac{1}{3}\int \frac{1}{\cos t}\cdot\frac{\cos t}{\sin t}\ dt = \frac{1}{3}\int \frac{1}{\sin t}\ dt =$ $\frac{1}{3}\int \csc t\ dt$. Now we can use formula 8 from the beginning of the exam. Then the integral equals $\ln|\csc t - \cot t| + C$. For the back-substitution, since $\tan t = x/3$, we have $\cot t = 3/x$. Setting up a trigonometric triangle, we have $\csc t = \frac{1}{\sin t} = \frac{\sqrt{x^2+9}}{x}$.

Thus $\int \frac{1}{x\sqrt{x^2+9}}\,dx = \frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}}{x} - \frac{3}{x}\right| + C$

V. Integrate using any applicable method or the table. If you use the table, give the number of the formula used.

A. (7.5) $\int e^{3x}\sqrt{1-e^{2x}}\,dx$.

Solution: We make the substitution $u = e^x,\ \ du = e^x\,dx$. Then $\int e^{3x}\sqrt{1-e^{2x}}\,dx =$ $\int e^x\cdot e^{2x}\sqrt{1-e^{2x}}\,dx = \int u^2\sqrt{1-u^2}\,du$. This is precisely formula 5 from the beginning of the exam. Thus, the integral equals
$\frac{u}{8}(2u^2-1)\sqrt{a^2-1} + \frac{1}{8}\arcsin u + C = \frac{e^x}{8}(2e^{2x}-1)\sqrt{a^2-1} + \frac{1}{8}\arcsin e^x + C.$

B. (7.5) $\int \frac{1}{x^2+4x+8}\,dx$

Solution: After completing the square in the denominator, we have
$\int \frac{1}{x^2+4x+8}\,dx = \int \frac{1}{(x+2)^2+4}\,dx$. After the substitution $u = x+2\ dx = du$, we can use formula 1 to obtain $\int \frac{1}{x^2+4x+8}\,dx = \frac{1}{2}\arctan\frac{x+2}{2} + C$.

VI. Decide if the following integral is convergent or divergent. If it is convergent, evaluate the integral.

A. (7.5)

$$\int_1^\infty \frac{1}{x\sqrt{x}}\,dx.$$

Solution: $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t x^{-3/2}\,dx = \lim_{t\to\infty}\left.\frac{x^{-1/2}}{-1/2}\right|_1^t = \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t}}+2\right) = 2.$

Thus the integral converges to 2.

B. (7.5)

$$\int_0^1 \frac{x}{1-x^2}\,dx.$$

Solution: The integrand has an infinite discontinuity at $x = 1$. Thus,
$\displaystyle\int_0^1 \frac{x}{1-x^2}\,dx = \lim_{t\to 1^-}\int_0^t \frac{x}{1-x^2}\,dx$. After the substitution $u = 1-x^2,\ \ d = -2x$, we find that an antiderivative of $\displaystyle\frac{x}{1-x^2}$ is $-\frac{1}{2}\ln|1-x^2| + C$. Then
$\displaystyle\int_0^1 \frac{x}{1-x^2}\,dx = \lim_{t\to 1^-}\left(-\frac{1}{2}\ln|1-t^2| + \frac{1}{2}\ln 1\right) = \infty$ (as $t \to 1^-$, we have $|1-t^2| \to 0^+$ and $\ln|1-t^2| \to -\infty$). Thus the integral diverges.