Mathematics 132 – Calculus for Physical and Life Sciences 2
Exam 3 – Review Sheet
April 16, 2009

Sample Exam Questions - Solutions

I. All parts of this question refer to the differential equation

$$y' = y(4 - y)$$

(1) Use Euler's method to approximate the solution of this equation with $y(0) = 1$ for $0 \leq x \leq 1$ using $n = 4$.

Solution: We have $\Delta x = 0.25$.

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = .25 & y_1 = y_0 + (y_0(4 - y_0))\Delta x = 1 + 3(.25) = 1.75 \\ x_2 = .5 & y_2 = y_1 + (y_1(4 - y_1))\Delta x = 2.734375 \\ x_3 = .75 & y_3 = y_2 + (y_2(4 - y_2))\Delta x = 3.599548340 \\ x_4 = 1 & y_4 = y_3 + (y_3(4 - y_3))\Delta x = 3.959909617 \end{array}$$

(2) This is a separable equation, find the general solution and determine the constant of integration from the initial condition $y(0) = 1$.

Solution: After separating the variables we have $\displaystyle\int \frac{1}{y(4-y)}\,dy = \int dx$. For the integral in y we use partial fractions: $\displaystyle\frac{1}{y(4-y)} = \frac{A}{y} + \frac{B}{4-y}$. We find that $A = B = 1/4$ and thus $\displaystyle\int \frac{1}{y(4-y)}\,dy = \frac{1}{4}\ln|y| - \frac{1}{4}\ln|4-y|$. Therefore, $\displaystyle\frac{1}{4}\ln\left|\frac{y}{4-y}\right| = x + C$. Then $\displaystyle\left|\frac{y}{4-y}\right| = e^{4x}\cdot e^{4C}$ and thus $\displaystyle\frac{y}{4-y} = A\cdot e^{4x}$. Solving for y, we obtain $\displaystyle y = \frac{4Ae^{4x}}{1 + Ae^{4x}}$.
The initial condition $y(0) = 1$ gives $\displaystyle 1 = \frac{4A}{1+A}$ and thus $A = 1/3$.

Therefore, the solution to the initial value problem is $\displaystyle y = \frac{4/3e^{4x}}{1 + 1/3e^{4x}}$.

II. Find the general solutions of the following differential equations

(1) $\displaystyle y' = \frac{y}{x(x+1)}$

Solution: This is a separable differential equation.

We have $\int \frac{dy}{y} = \int \frac{dx}{x(x+1)}$. For the integral on the right we use partial fractions: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

Thus $\int \frac{1}{x(x+1)}\, dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$.

We have $\ln|y| = \ln\left|\frac{x}{x+1}\right| + C$ and thus $|y| = e^{\ln\left|\frac{x}{x+1}\right| + C} = \left|\frac{x}{x+1}\right| \cdot e^C$.

Therefore $y = A\frac{x}{x+1}$ is the general solution of the given differential equation.

(2) $y' = \frac{\sqrt{1-x^2}}{e^{2y}}$.

Solution: This is a separable differential equation.

We have $\int e^{2y}\, dy = \int \sqrt{1-x^2}\, dx$. For the integral on the right we use the trigonometric substitution $x = \sin\theta$, $dx = \cos\theta\, d\theta$. Thus $\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta =$ $\int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C =$

$\frac{1}{2}\theta + \frac{1}{4}2\sin\theta\cos\theta + C = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

Therefore $\frac{1}{2}e^{2y} = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$ or $e^{2y} = \arcsin x + x\sqrt{1-x^2} + D$ and we have that $y = \frac{1}{2}\ln(\arcsin x + x\sqrt{1-x^2} + D)$ is the general solution to the given differential equation.

III. Cesium 137 decays at a rate proportional to the amount of itself present and has a half-life of 30 years. Suppose we have a sample of 100 mg. of Cesium 137.

(1) Write a differential equation whose solution is $y(t)$, the amount of Cesium 137 left after t years.

Solution: $\frac{dy}{dx} = ky$. To determine the constant k we have to solve the equation first.

(2) Give a formula for the amount $y(t)$ of Cesium 137 left after t years.

Solution: Since the differential equation is an exponential decay equation, we know from class that the general solution is of the form $y(t) = Ce^{kt}$. Since $y(0) = 100$, we have $C = 100$ and thus $y(t) = 100e^{kt}$. Since the half-life is 30 years, we have $y(30) = 50$ and thus $50 = 100e^{k\cdot 30}$. Solving for k, we obtain $k = \frac{1}{30}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{30}$.

Therefore, the amount of Cesium 137 left after t years is given by

$$y(t) = 100e^{-\frac{\ln(2)t}{30}}.$$

We can also revise our answer to part (1) using the value of k we just obtained. The differential equation is

$$\frac{dy}{dx} = -\frac{\ln(2)}{30}y.$$

(3) After 98 years, how much Cesium 137 is left of the sample?

Solution: $y(98) = 100e^{-\frac{\ln(2)98}{30}} \approx 10.39$ mg.

(4) How long will it take until only 1 mg of the sample is left?

Solution: Find t when $y(t) = 1$. We need to solve the equation $100e^{-\frac{\ln(2)t}{30}} = 1$. We have

$$\begin{aligned} e^{-\frac{\ln(2)t}{30}} &= \frac{1}{100} \\ -\frac{\ln(2)t}{30} &= \ln(1/100) = \ln(100) \\ \frac{\ln(2)t}{30} &= \ln(100) \\ t = \frac{30\ln(100)}{\ln(2)} &\approx 199.32 \text{ years.} \end{aligned}$$

IV.

(1) Does the sequence $a_n = n\ln(1+n)$ converge? Why or why not? Does the infinite series $\sum_{n=1}^{\infty} n\ln(1+n)$ converge? Why or why not?

Solution: The sequence $a_n = n\ln(1+n)$ is not bounded and thus it does not converge. Since $\lim_{n\to\infty} n\ln(1+n) \neq 0$, the series $\sum_{n=1}^{\infty} n\ln(1+n)$ diverges (by the Divergence Test).

(2) Use the Integral Test to determine whether or not

$$\sum_{k=1}^{\infty} \frac{k}{e^k}$$

converges.

Solution: The function $f(x) = \dfrac{x}{e^x}$ is continuous and positive. Since $f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{e^x(1-x)}{e^{2x}} < 0$ for $x > 1$, f(x) is also decreasing for $x > 1$.

Consider $\displaystyle\int_1^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx$. Using integration by parts, $u = x$, $du = dx$, $dv = e^{-x}dx$, $v = -e^{-x}$, the improper integral equals $\displaystyle\lim_{b\to\infty}\left(-be^{-b} + e^{-1} + \int_1^b e^{-x}\,dx\right) = \lim_{b\to\infty}\left(-be^{-b} + e^{-1} - e^{-b} + e^{-1}\right)$. Since $\displaystyle\lim_{b\to\infty} e^{-b} = 0$ and $\displaystyle\lim_{b\to\infty} be^{-b} = \lim_{b\to\infty}\frac{b}{e^b} \overset{l'H}{=} \lim_{b\to\infty}\frac{1}{e^b} = 0$, the improper integral converges to $2e^{-1}$. By the Integral Test, the series $\displaystyle\sum_{k=1}^\infty \frac{k}{e^k}$ converges.

(3) Use the Ratio Test to determine whether or not

$$\sum_{k=0}^\infty \frac{3^n}{n!}$$

converges.

Solution:

$$\lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty}\frac{3}{n+1} = 0 < 1.$$

By the Ratio Test, the series converges.

(4) Determine (with justification!) whether or not the following series converge:

$$\sum_{k=1}^\infty \frac{1}{\sqrt{k}}, \qquad \sum_{n=0}^\infty (-1)^n\frac{3^n}{\pi^{2n}}, \qquad \sum_{n=1}^\infty \frac{1}{n^{1.01}}.$$

Solution: The series $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{k}}$ is the p-series with $p = 1/2$ and thus it diverges.

The series $\displaystyle\sum_{n=0}^\infty (-1)^n\frac{3^n}{\pi^{2n}}$ is the geometric series with ratio $\dfrac{-3}{\pi^2}$. Since the ratio is less than 1 in absolute value, the series converges. (The sum of the series is $\dfrac{1}{1+\frac{3}{\pi^2}}$.)

The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{1.01}}$ is the p-series with $p = 1.01$. Since $p > 1$, the p-series converges.

V. For each of the given power series, find the interval of convergence.

$$f(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}, \qquad g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}.$$

(In particular, give the radius of convergence, and investigate convergence at the endpoints.)

Solution: For $f(x) = \sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$, consider the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+1}}{\sqrt{n+1}}}{\frac{(2x)^n}{\sqrt{n}}} \right| = \lim_{n\to\infty} 2|x| \frac{\sqrt{n}}{\sqrt{n+1}} = 2|x|.$$

The series converges if $|x| < 1/2$ and it diverges if $|x| > 1/2$. Since the series is centered at 0 the radius of convergence is $1/2$.

If $x = 1/2$, the series equals $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is the p-series with $p = 1/2$. Since $p < 1$, the series diverges.

If $x = -1/2$, the series equals $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Since the sequence $\frac{1}{\sqrt{n}}$ is decreasing and it converges to 0 as $b \to \infty$, the series converges by the Alternating Series Test.

The interval of convergence for the first series is $[-1/2, 1/2)$.

We consider the Ratio Test for $g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-5)^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{\frac{|x-5|^{n+1}}{(n+1)3^{n+1}}}{\frac{|x-5|^n}{n \cdot 3^n}} = \lim_{n\to\infty} \frac{|x-5| \cdot n}{3(n+1)} = \frac{|x-5|}{3}.$$

The series converges if $|x-5| < 3$ and it diverges if $|x-5| > 3$. Thus the radius of convergence is 3.

If $x - 5 = 3$, *i.e.*, $x = 8$, the series becomes the alternating harmonic series and it converges.

If $x - 5 = -3$, *i.e.*, $x = 2$, the series equals $g(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is the negative of the harmonic series and thus it diverges.

The interval of convergence for the second series is $(2, 8]$.