Mathematics 132 – Calculus for Physical and Life Sciences 2
Exam 2 – Review Sheet - Solutions - Part 2
March 13, 2009

VII.

(A) Find the center of mass of a thin rod 7 yards long if the mass density x yards from the left end of the rod equals $\rho(x) = x\ln(x+1)$ Kg/yd.

Solution: Place the rod on the x-axis with the left end at the origin. Then the center of mass of the rod is situated at

$$\bar{x} = \frac{\int_0^7 x\rho(x)\ dx}{\int_0^7 \rho(x)\ dx} = \frac{\int_0^7 x^2\ln(x+1)\ dx}{\int_0^7 x\ln(x+1)\ dx}.$$

For $\displaystyle\int_0^7 x^2\ln(x+1)\ dx$ we use integration by parts with

$$u = \ln(x+1),\ u' = \frac{1}{x+1} \text{ and } v' = x^2,\ v = \frac{x^3}{3}.$$

Thus

$$\int_0^7 x^2\ln(x+1)\ dx = \frac{x^3}{3}\ln(x+1)\Big|_0^7 - \frac{1}{3}\int_0^7 \frac{x^3}{x+1}\ dx.$$

For the last integral we perform long division:

$$\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}.$$

Therefore,

$$\int_0^7 x^2\ln(x+1)\ dx = \frac{x^3}{3}\ln(x+1)\Big|_0^7 - \frac{1}{3}\left(\frac{x^3}{3} - \frac{x^2}{2} + x - \ln(x+1)\right)\Big|_0^7 =$$

$$= \frac{7^3}{3}\ln 8 - \frac{1}{3}\left(\frac{7^3}{3} - \frac{7^2}{2} + 7 - \ln 8\right) \approx 206.165$$

For $\displaystyle\int_0^7 x\ln(x+1)\ dx$ we use integration by parts with

$$u = \ln(x+1),\ u' = \frac{1}{x+1} \text{ and } v' = x,\ v = \frac{x^2}{2}.$$

Thus

$$\int_0^7 x\ln(x+1)\ dx = \frac{x^2}{2}\ln(x+1)\Big|_0^7 - \frac{1}{2}\int_0^7 \frac{x^2}{x+1}\ dx.$$

For the last integral we perform long division:

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}.$$

Therefore,

$$\int_0^7 x\ln(x+1)\ dx = \frac{x^2}{2}\ln(x+1)\Big|_0^7 - \frac{1}{2}\left(\frac{x^2}{2} - x + \ln(x+1)\right)\Big|_0^7 =$$

$$\frac{7^2}{2}\ln 8 - \frac{1}{2}\left(\frac{7^2}{2} - 7 + \ln 8\right) \approx 41.1566$$

Thus

$$\bar{x} \approx \frac{206.165}{41.1566} \approx 5.00928$$

(B) Find the center of mass of a this plate in the shape of the region bounded by the curves $f(x) = e^x$, $x = 0$, $x = 1$ and $y = 0$

Solution: The center of mass of the plate has coordinates

$$\bar{x} = \frac{\int_0^1 xe^x\ dx}{\int_0^1 e^x\ dx}, \quad \bar{y} = \frac{\frac{1}{2}\int_0^1 e^{2x}\ dx}{\int_0^1 e^x\ dx}$$

Using integration by parts with $u = x$, $u' = 1$ and $v' = e^x$, $v = e^x$,

$$\int_0^1 xe^x\ dx = xe^x|_0^1 - \int_0^1 e^x\ dx = e - e^x|_0^1 = e - (e-1) = 1.$$

We have $\displaystyle\int_0^1 e^x\ dx = e - 1$ and $\displaystyle\int_0^1 e^{2x}\ dx = \frac{1}{2}\ e^{2x}\Big|_0^1 = \frac{1}{2}(e^2 - 1)$. Thus

$$\bar{x} = \frac{1}{e-1} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{4}(e^2-1)}{e-1}$$

VIII. Consider the differential equation $y' = x^2(y+1)$.

(A) Verify that every member of the family of functions

$$y = Ce^{\left(\frac{x^3}{3}\right)} - 1$$

is a solution for the differential equation above.

Solution: We have LHS$=y' = Ce^{\left(\frac{x^3}{3}\right)} \cdot x^2$ and RHS$=x^2(y+1) = x^2 \cdot (Ce^{\left(\frac{x^3}{3}\right)} - 1 + 1) = x^2 \cdot Ce^{\left(\frac{x^3}{3}\right)}$. SInce $LHS = RHS$ each member of the given family is a solution to the differential equation.

(B) Find the solution to differential equation above which also satisfies $y(0) = -2$.

Solution: A function from the family in part (A) must satisfy the additional condition $y(0) = -2$. Thus $-2 = C - 1$ and we must have $C = -1$. Therefore, the solution to the given initial value problem is $y = -e^{\left(\frac{x^3}{3}\right)} - 1$.

IX. Match the following differential equations with the slope field (direction field) below. Note that there are only 3 plots, so the correct answer for one equation is "None". Circle the correct answers.

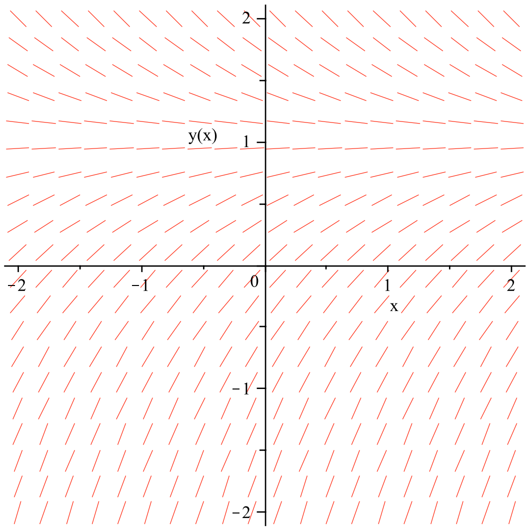
(1) $y' = x$ (I) (II) (III) None

(2) $y' = 1 + y^2$ (I) (II) (III) None

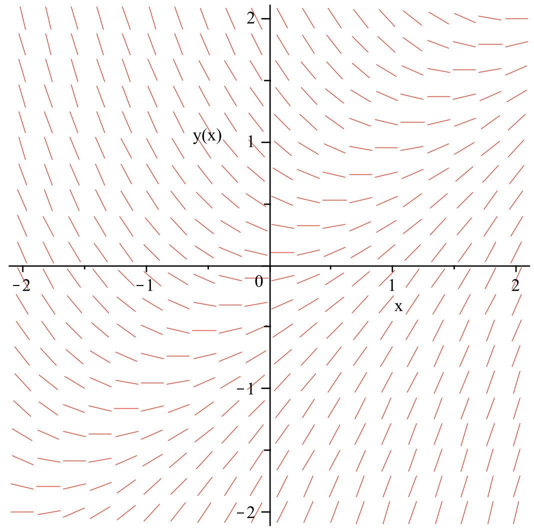
(3) $y' = 1 - y$ (I) (II) (III) None

(4) $y' = x - y$ (I) (II) (III) None

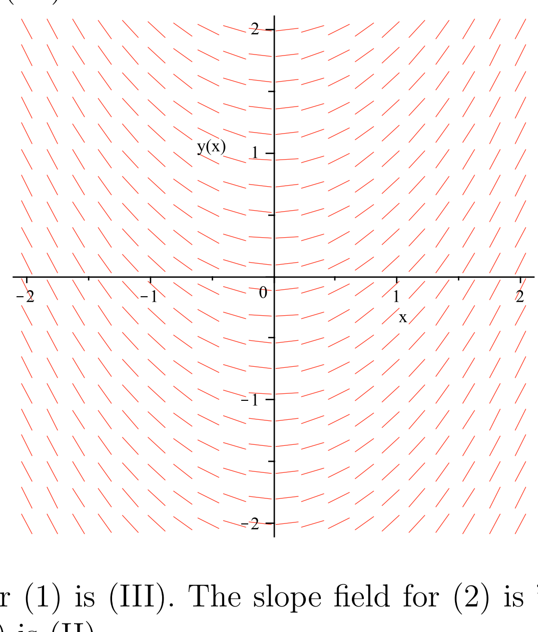
(I)



(II)



(III)



Solution: The slope field for (1) is (III). The slope field for (2) is "None". The slope field for (3) is (I). The slope field for (4) is (II).