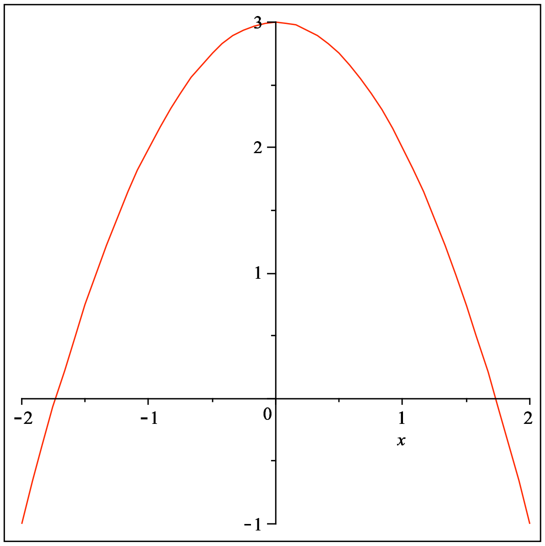
III.

(A) Let R be the region in the plane bounded by $y = 3 - x^2$ and the x-axis.

(1) Sketch the region R.

Solution: The region is bounded above by the parabola $y = 3 - x^2$, and below by the x-axis. It extends from $x = -\sqrt{3}$ to $x = \sqrt{3}$:



(2) Find the area of R.

Solution: The area is

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} 3 - x^2 \, dx = 2\int_{0}^{\sqrt{3}} 3 - x^2 \, dx = 2\left(3x - \frac{x^3}{3}\Big|_0^{\sqrt{3}} \right) = 4\sqrt{3}.$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

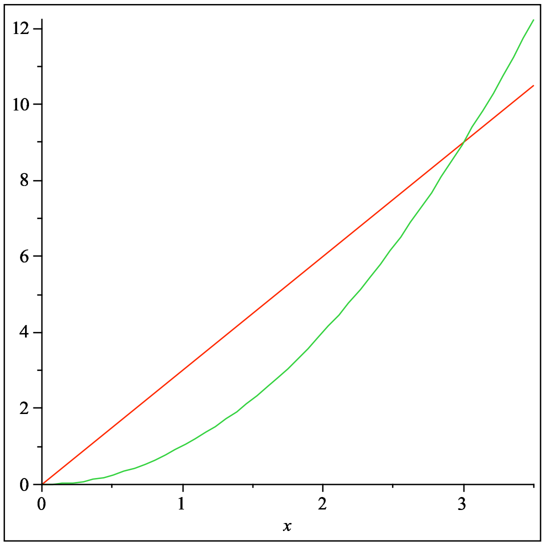
Solution: Rotating R around the x-axis, the cross-sections are disks with radius $y = 3 - x^2$, so

$$\begin{aligned} V &= \int_{-\sqrt{3}}^{\sqrt{3}} \pi(3 - x^2)^2 \, dx \\ &= 2\pi \int_0^{\sqrt{3}} 9 - 6x^2 + x^4 \, dx \\ &= 2\pi \left(9x - 2x^3 + \frac{x^5}{5}\Big|_0^{\sqrt{3}} \right) \\ &= \frac{48\pi\sqrt{3}}{5}. \end{aligned}$$

(B) Let R be the region in the plane bounded by $y = 3x$ and $y = x^2$.

(1) Sketch the region R.

Solution: The region is bounded above by the line $y = 3x$ and below by the parabola $y = x^2$. It extends from $x = 0$ to $x = 3$:



(2) Find the area of R.

Solution: The area is

$$A = \int_0^3 3x - x^2\, dx = \left(\frac{3x^2}{2} - \frac{x^3}{3} \Big|_0^3 \right) = \frac{9}{2}.$$

(3) Find the volume of the solid generated by rotating R about the x-axis. *Solution:* The cross-sections by planes perpendicular to the x-axis are washers with inner radius $r_{in} = x^2$ and outer radius $r_{out} = 3x$. The volume is

$$\begin{aligned} V &= = \int_0^3 \pi(3x)^2 - \pi(x^2)^2\, dx \\ &= \pi \int_0^3 9x^2 - x^4\, dx \\ &= \pi \left(3x^3 - \frac{x^5}{5} \Big|_0^3 \right) \\ &= \frac{162\pi}{5}. \end{aligned}$$

(4) Find the volume of the solid generated by rotating R about the y-axis.

Solution: For rotating about the y-axis, we slice horizontally and set up the integral in terms of y. The horizontal cross-sections are washers too, with inner radius $r_{in} = y/3$ (from the line), and outer radius $r_{out} = \sqrt{y}$ (from the parabola).

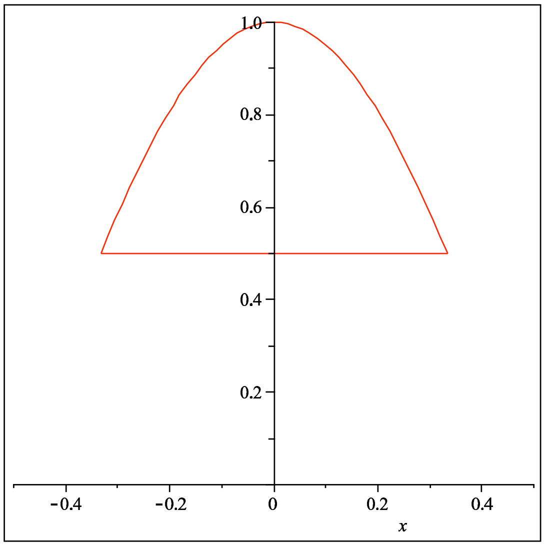
The solid extends from $y = 0$ to $y = 9$ along the y-axis. This gives

$$\begin{aligned} V &= = \int_0^9 \pi(\sqrt{y})^2 - \pi(y/3)^2 \, dy \\ &= \pi \int_0^9 y - \frac{y^2}{9} \, dy \\ &= \pi \left(\frac{y^2}{2} - \frac{y^3}{27} \Big|_0^9 \right) \\ &= \frac{27\pi}{2}. \end{aligned}$$

(C) Let R be the region in the plane bounded by $y = \cos(\pi x)$, $y = 1/2$, $x = -1/3$ and $x = 1/3$.

(1) Sketch the region R.

Solution: The region is bounded above by an arc of the cosine graph and below by the line $y = 1/2$:



(2) Find the area of R.

Solution: The area is (using symmetry):

$$\begin{aligned} A &= \int_{-1/3}^{1/3} \cos(\pi x) - 1/2 \, dx \\ &= 2 \int_0^{1/2} \cos(\pi x) - 1/2 \, dx \\ &= 2 \left(\frac{1}{\pi} \sin(\pi x) - \frac{x}{2} \Big|_0^{1/3} \right) \\ &= \frac{\sqrt{3}}{\pi} - \frac{1}{3}. \end{aligned}$$

(3) Find the volume of the solid generated by rotating R about the x-axis.

Solution: The cross-sections by planes perpendicular to the x-axis are washers with inner radius $r_{in} = 1/2$ and outer radius $r_{out} = \cos(\pi x)$. By symmetry (and integrating with the 1/2-angle formula for $\cos^2(\pi x)$), our volume will be

$$\begin{aligned} V &= \int_{-1/3}^{1/3} \pi \cos(\pi x)^2 - \pi \left(\frac{1}{2}\right)^2 \, dx \\ &= 2\pi \int_0^{1/3} \frac{1}{2}(1 + \cos(2\pi x)) - \frac{1}{4} \, dx \\ &= 2\pi \left(\frac{x}{4} + \frac{1}{4\pi} \sin(2\pi x) \Big|_0^{1/3} \right) \\ &= \frac{\sqrt{3}}{4} + \frac{\pi}{6}. \end{aligned}$$

IV. The height of a monument is 20m. The horizontal cross-section of the monument at x meters from the top is an isosceles right triangle with legs $x/4$ meters. Find the volume of the monument.

Solution: The area of the cross-section x meters from the top is $A(x) = \frac{1}{2}\left(\frac{x}{4}\right)^2 = \frac{x^2}{32}$. So the volume is

$$V = \int_0^{20} \frac{x^2}{32} \, dx = \left(\frac{x^3}{96} \Big|_0^{20} \right) = \frac{250}{3}.$$

cubic meters.

V.

(A) Set up and evaluate the integral to compute the arclength of the curve $x = 3t^2$, $y = 2t^3$, $0 \le t \le 2$.

Solution: By the arclength integral formula,

$$\begin{aligned} L &= \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \\ &= \int_0^2 \sqrt{(6t)^2 + (6t^2)^2} \, dt \\ &= \int_0^2 6t\sqrt{1 + t^2} \, dt. \end{aligned}$$

Now make a u-substitution $u = 1 + t^2$, so $du = 2t\ dt$. The integral converts to

$$
\begin{aligned}
&= \int_{u=1}^{u=5} 3\sqrt{u}\ du \\
&= \left(3 \cdot \frac{2}{3} u^{3/2} \Big|_1^5 \right) \\
&= 2(5\sqrt{5} - 1).
\end{aligned}
$$

(B) Set up and evaluate the integral to compute the arclength of the curve $y = \frac{1}{6}(x^2+4)^{3/2}$, $0 \leq x \leq 3$. (Hint: the arclength integral simplifies to a manageable form if you are careful with the algebra.)

Solution: By the arclength integral formula, we must integrate

$$
\begin{aligned}
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{4}(x^2+4)^{1/2}(2x)\right)^2} \\
&= \sqrt{1 + \left(\frac{x}{2}(x^2+4)^{1/2}\right)^2} \\
&= \sqrt{1 + x^2 + \frac{x^4}{4}} \\
&= \sqrt{\left(1 + \frac{x^2}{2}\right)^2} \\
&= 1 + \frac{x^2}{2}.
\end{aligned}
$$

(In other words, $1 + \left(\frac{dy}{dx}\right)^2$ is itself a perfect square in this case!) Hence

$$
L = \int_0^3 1 + \frac{x^2}{2}\ dx = \left(x + \frac{x^3}{6} \Big|_0^3 \right) = \frac{15}{2}.
$$

VI.

(A) Find the average value of $f(x) = \sqrt{1 - x^2}$ on the interval $[0, 1/2]$. (Use trigonometric substitution, not the table.)

Solution: The average value is

$$\begin{aligned}
f_{ave} &= \frac{1}{1/2-0}\int_0^{1/2} \sqrt{1-x^2}\,dx \\
&= 2\int_0^{\pi/6} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta \text{ (letting } x=\sin\theta, dx=\cos\theta\,d\theta) \\
&= 2\int_0^{\pi/6} \cos^2\theta\,d\theta \\
&= 2\int_0^{\pi/6} \frac{1}{2}(1+\cos(2\theta))\,d\theta \text{ (half-angle formula)} \\
&= \left(\theta+\frac{1}{2}\sin(2\theta)\Big|_0^{\pi/6}\right) \\
&= \frac{\pi}{6}+\frac{\sqrt{3}}{4}.
\end{aligned}$$

(B) Find the average value of $f(x) = x\sqrt{1+x^4}$ on the interval $[0,2]$.

Solution: The average value is

$$\begin{aligned}
f_{ave} &= \frac{1}{2-0}\int_0^{2} x\sqrt{1+x^4}\,dx \\
&= \frac{1}{4}\int_{u=0}^{u=4} \sqrt{1+u^2}\,du \text{ (letting } u=x^2, du=2x\,dx) \\
&= \left(\frac{u}{8}\sqrt{1+u^2}+\frac{1}{8}\ln(u+\sqrt{1+u^2})\Big|_0^{4}\right) \text{ (by \# 21 in table)} \\
&= \frac{\sqrt{17}}{2}+\frac{1}{8}\ln(4+\sqrt{17}).
\end{aligned}$$