College of the Holy Cross, Spring 2009
Math 132, Midterm Exam 3
Solutions

I. A. (15 points) Solve the initial value problems

(a) $\dfrac{dP}{dt} = 0.02P, \quad P(0) = 20$

Solution: This is a differential equation for exponential growth. The general solution is $P(t) = Ae^{0.02t}$. Since $P(0) = 20$, we have $20 = P(0) = A$. Thus, the solution to the initial value problem is $P(t) = 20e^{0.02t}$.

(b) $\dfrac{dy}{dx} = e^{x-y}, \quad y(0) = 1$

Solution: Rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{e^x}{e^y}$. After separating the variables, we have: $\displaystyle\int e^y\,dy = \int e^x\,dx \iff e^y = e^x + C \iff y = \ln(e^x + C)$.

Warning: $\ln(e^x + C) \neq \ln(e^x) + \ln(C)$.

Using the initial condition $y(0) = 1$, we have $1 = \ln(1 + C)$, *i.e.*, $1 + C = e$. Thus, $C = e - 1$ and the solution to the initial value problem is $y(x) = \ln(e^x + e - 1)$.

(c) $\dfrac{dy}{dx} = \dfrac{5}{x} - \dfrac{y}{2x}, \quad$ y(1)=3

Solution: Rewrite the differential equation as $\dfrac{dy}{dx} = \dfrac{10-y}{2x}$. After separating the variables, we have: $\displaystyle\int \frac{dy}{10-y} = \int \frac{dx}{2x} \iff -\ln(10-y) = \frac{1}{2}\ln(2x) + C \iff$
$\ln(10-y) = -\frac{1}{2}\ln(2x) + D \iff \ln(10-y) = \ln(2x)^{-1/2} + D \iff$
$|10-y| = (2x)^{-1/2}\cdot e^D \iff 10-y = A(2x)^{-1/2} \iff 10-y = A\frac{1}{\sqrt{2x}} \iff y = 10 - A\frac{1}{\sqrt{2x}}$.
Using the initial condition $y(1) = 3$, we have $3 = 10 - A$, *i.e.*, $A - 7$. The solution to the initia value problem is $y = 10 - \dfrac{7}{\sqrt{2x}}$.

B. (7 points) For the initial value problem in part A.a., *i.e.*,

$$\frac{dP}{dt} = 0.02P, \quad P(0) = 20,$$

use Euler's method with **two** steps to approximate the value of $P(1)$

Solution: Since we need to get from 0 to 1 in two steps, $h = \Delta t = 0.5$.

n	t	P	$P' = 0.02P$	$\Delta P = P' \cdot \Delta t$
0	0	20	$0.02 \cdot 20 = 0.4$	$0.4 \cdot 0.5 = 0.2$
1	0.5	20.2	$0.02 \cdot 20.2 = 0.404$	$0.404 \cdot 0.5 = 0.202$
2	1	20.402		

Thus, $P(1) \approx 20.402$.

II. (20 points) A bacteria population triples in size every two days. Suppose we start with a population of 250 bacteria. Let $P(t)$ denote the number of bacteria after t days.

(a) How many bacteria are there after 8 days?

Solution: $P(0) = 250$, $P(2) = 3 \cdot 250 = 750$, $P(4) = 3 \cdot 750 = 2250$, $P(6) = 3 \cdot 2250 = 6750$ and $P(8) = 3 \cdot 6750 = 20250$

(b) Set up a differential equation whose solution is $P(t)$

Solution: This is an exponential growth model. Thus, $\frac{dP}{dt} = kP$.

(c) Solve the differential equation in part (b) to find a formula for $P(t)$.

Solution: The solution to the exponential growth differential equation is $P(t) = Ae^{kt}$. (You can separate the variables and solve the equation. This was done in class.) Since $P(0) = 250$, we have $A = 250$. Since $P(8) = 20250$, we have $20250 = 250e^{k \cdot 8}$, *i.e.*, $e^{8k} = 81$ and thus $k = \frac{1}{8}\ln(81) = \frac{1}{8}\ln(3^4) = \frac{1}{2}\ln(3)$. The solution to this initial value problem is $P(t) = 250e^{k\frac{1}{2}\ln(3)t}$ or $P(t) = 250 \cdot 3^{t/2}$. The differential equation from part (b) is $\frac{dP}{dt} = \frac{1}{2}\ln(3)P$.

(d) When will the population reach 50,000 bacteria?

Solution: We solve the equation $50,000 = 250 \cdot 3^{t/2}$. We have: $3^{t/2} = 200 \iff \frac{t}{2}\ln(3) = \ln(200) \iff t = 2 \cdot \frac{\ln(200)}{\ln(3)} \approx 9.63$ days.

III. (15 points) For each sequence $\{a_n\}$ with general term given below, decide if the sequence converges or diverges. If it converges, find its limit. Show your work!

(a) $a_n = \frac{3 - 2n^2}{2 - 3n^2}$ **Solution:** Using the rule of l'Hospital twice (or the fact that both numerator and denominator are polynomials of degree 2), we have $\lim_{n\to\infty} \frac{3 - 2n^2}{2 - 3n^2} = \frac{2}{3}$. Thus, the sequence is convergent to $\frac{2}{3}$.

(b) $a_n = \frac{(n+1)!}{n!}$ **Solution:** Since $\frac{(n+1)!}{n!} = n + 1$, $\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$ and the sequence diverges.

(c) $\frac{\sin n + 1}{n^2 + 1}$ **Solution:** The numerator is not convergent and the denominator tends to ∞. We will use the squeeze theorem: $-1 \leq \sin(n) \leq 1 \iff 0 \leq \sin(n) + 1 \leq 2 \iff 0 \leq \frac{\sin n + 1}{n^2 + 1} \leq \frac{2}{n^2 + 1}$. Since $\lim_{n\to\infty} \frac{2}{n^2 + 1} = 0$, we have $\lim_{n\to\infty} \frac{\sin n + 1}{n^2 + 1} = 0$. The sequence converges to 0.

IV. (10 points) For each of the series below, decide if the series converges or diverges. If it converges, find its sum. If it diverges, explain why if diverges.

(a) $\sum_{n=0}^{\infty} \frac{2\pi^n}{4^n}$ **Solution:** We have $\sum_{n=0}^{\infty} \frac{2\pi^n}{4^n} = \sum_{n=0}^{\infty} 2\left(\frac{\pi}{4}\right)^n$. This is a geometric series with ratio equal to $\frac{\pi}{4}$, which is less than 1 in absolute value. The series converges and the sum is equal to $\frac{2}{1-\frac{\pi}{4}}$.

(b) $\sum_{n=0}^{\infty} \frac{2 \cdot 4^n}{\pi^n}$ **Solution:** We have $\sum_{n=0}^{\infty} \frac{24^n}{\pi^n} = \sum_{n=0}^{\infty} 2\left(\frac{4}{\pi}\right)^n$. This is a geometric series with ratio equal to $\frac{4}{\pi}$, which is greater than 1. The series diverges.

V. (20 points) For each of the series below, decide if the series converges or diverges. Clearly explain your decision and state which test of convergence you are using.

(a) $\sum_{n=0}^{\infty} \frac{1}{3n^5+1}$ **Solution:** We use the comparison test. As $n \to \infty$, $\frac{1}{3n^5+1}$ behaves like $\frac{1}{3n^5}$ and $\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^5}$ is convergent (p-series with $p = 5$. Since $\frac{1}{3n^5+1} \le \frac{1}{3n^5}$, the series $\sum_{n=0}^{\infty} \frac{1}{3n^5+1}$ converges as well.

(b) $\sum_{n=0}^{\infty} \frac{n^4}{3n^5+1}$ **Solution:** As $n \to \infty$, $\frac{n^4}{3n^5+1}$ behaves like $\frac{n^4}{3n^5} = \frac{1}{3n}$ and $\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (a multiple of the harmonic series). However, we cannot establish the inequality $\frac{n^4}{3n^5+1} \ge \frac{1}{3n}$. We use the limit comparison test instead.

$$\lim_{n\to\infty} \frac{\frac{n^4}{3n^5+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^5}{3n^5+1} = \frac{1}{3} \neq 0.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=0}^{\infty} \frac{n^4}{3n^5+1}$.

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{1}{3n+2}$ **Solution:** We use the alternating series test. The sequence $b_n = \frac{1}{3n+2}$ is decreasing and it converges to 0 as $n \to \infty$. Thus, the series converges.

(d) $\sum_{n=0}^{\infty} (-1)^n \frac{n}{3n+2}$ **Solution:** Since $\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} \neq 0$, the limit of terms, $\lim_{n\to\infty} (-1)^n \frac{n}{3n+2}$ does not exist. In particular, it does not equal zero. By the divergence test, the series diverges.

VI. (13 points) Consider the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x+4)^n}{n \cdot 4^n}$.

(a) Where is this series centered? **Solution:** The center of the series is $a = -4$.

(b) Find the radius of convergence of the series. **Solution:** $\lim_{n\to\infty} \left| \frac{\frac{(x+4)^{(n+1)}}{(n+1)\cdot 4^{(n+1)}}}{\frac{(x+4)^n}{n \cdot 4^n}} \right| = \lim_{n\to\infty} \frac{n|x+4|}{4(n+1)} = \frac{1}{4}|x+4|$. The radius of convergence is $R = 4$.

(c) Find the interval of convergence of the series (don't forget to check the endpoints of the interval).

Solution: For the interval of convergence, we solve the inequality $|x+4| < 4$:

$$-4 < x + 4 < 4$$

$$-8 < x < 0$$

The series is convergent in the interval $(-8, 0)$ and divergent outside this interval. Now we check for convergence at the end points of the interval.

If $x = -8$ the series becomes $\sum_{n=1}^{\infty}(-1)^n \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty}(-1)^n \frac{(-1)^n 4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is the harmonic series and it diverges.

If $x = 0$ the series becomes $\sum_{n=1}^{\infty}(-1)^n \frac{4^n}{n4^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$. This is the alternating harmonic series and it converges.

Thus, the interval of convergence is $(-8, 0]$.