College of the Holy Cross, Spring 2009
Math 132, Midterm Exam 2
Solutions

Formulas that may be useful:

$\cos^2(t) = \dfrac{1+\cos(2t)}{2}$, $\sin^2(t) = \dfrac{1-\cos(2t)}{2}$

1. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$

2. $\displaystyle\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$

3. $\displaystyle\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln|u+\sqrt{u^2+a^2}| + C$

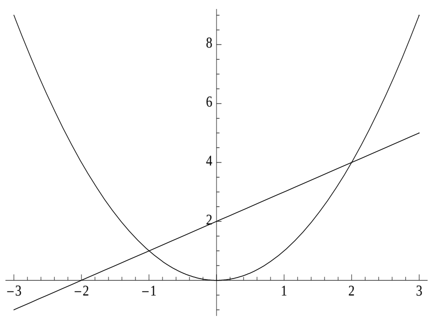
4. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C$

5. $\displaystyle\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\frac{u}{a} + C$

6. $\displaystyle\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arccos\left(\frac{a-u}{a}\right) + C$

I. (10) The region R is bounded by the curves $y = x+2$, $y = x^2$, $x = -2$ and $x = 2$. Sketch the region R and find its area.

Solution:



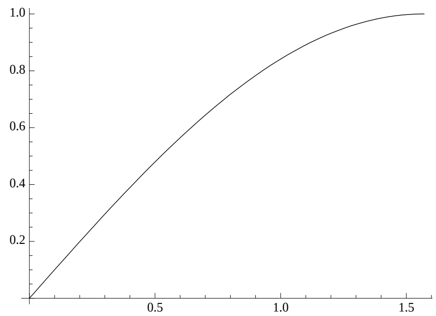
The curves meet when $x^2 = x+2$, or when $x^2 - x - 2 = 0$. The solutions are $x = -1$ and $x = 2$. Thus the area equals

$$\int_{-2}^{-1} x^2-(x+2)\,dx + \int_{-1}^{2} x+2-x^2\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Big|_{-2}^{-1} + \left(\frac{x^2}{2} + 2x - \frac{x^3}{3}\right)\Big|_{-1}^{2} = \frac{11}{6} + \frac{9}{2} = \frac{19}{3}.$$

II. The region R is bounded by the graphs of $y = \sin x$, $x = 0$, $x = \dfrac{\pi}{2}$ and the x-axis.

A. (10) The region R is rotated about the x-axis to generate a solid. Set up and compute the integral to find its volume.

Solution:

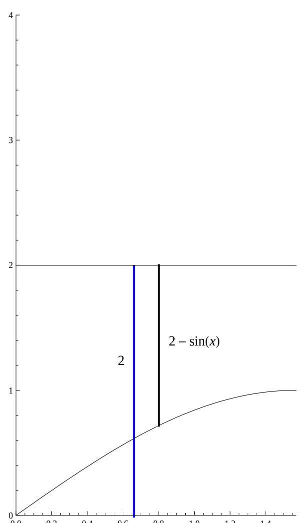


Volume$= \pi \displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^2 \; dx = \pi \int_0^{\frac{\pi}{2}} \frac{1 - \cos(2x)}{2} \; dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} 1 - \cos(2x) \; dx =$

$$= \frac{\pi}{2}\left(x - \frac{1}{2}\sin(2x)\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}\left(\frac{\pi}{2} - \frac{1}{2}\sin(\pi) - 0 + \frac{1}{2}\sin(0)\right) = \frac{\pi^2}{4}.$$

B. (10) The region R is rotated about the line $y = 2$ to generate a solid. Set up and compute the integral to find its volume.

Solution:

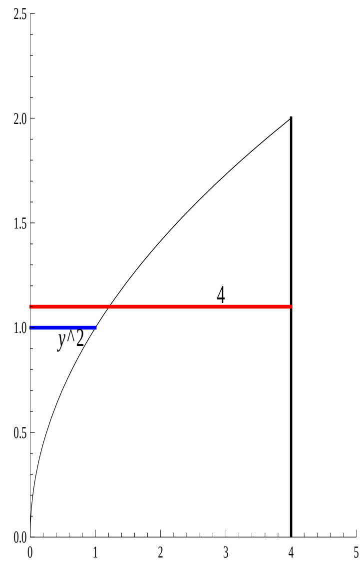


The slice is a washer with the big radius equal to 2 and the small radius equal to $2 - \sin x$.

Volume$= \pi \displaystyle\int_0^{\frac{\pi}{2}} 2^2 - (2 - \sin x)^2 \; dx = \pi \int_0^{\frac{\pi}{2}} 4 - 4 + 4\sin x - (\sin x)^2 \; dx = 4\pi \int_0^{\frac{\pi}{2}} \sin x \; dx -$ $\pi \int_0^{\frac{\pi}{2}} (\sin x)^2 \; dx$. We already calculated the second integral in part A., so the volume equals $4\pi\left(- \cos x\right)\Big|_0^{\frac{\pi}{2}} - \dfrac{\pi^2}{4} = 4\pi - \dfrac{\pi^2}{4}$.

III. (10) The region R is bounded by the graphs of $y = \sqrt{x}$, $x = 0$, $x = 4$ and the x-axis. The region R is rotated about the y-axis to generate a solid. Set up and compute the integral to find its volume.

Solution:



Since we are rotating with respect to the y-axis, we will obtain an integral with respect to y. Thus, we rewrite $y = \sqrt{x}$ as $x = y^2$. The cross section is a washer with the big radius equal to 4 and the small radius equal to y^2.

$$\text{Volume}=\pi \int_0^2 4^2 - (y^2)^2 \; dx = \pi \left(16y - \frac{y^5}{5}\right)\bigg|_0^2 = \frac{128}{5}\pi.$$

IV. (15) Set up and compute the integral for the arclength of the curve $y = \sqrt{x^3}$, $0 \le x \le 2$. If you prefer, you can think of the curve as the parametric curve $x = t$, $y = \sqrt{t^3}$, $0 \le t \le 2$.

Solution:

$$\text{Arclength} = \int_0^2 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2} \; dx = \int_0^2 \sqrt{1 + \frac{9}{4}x} \; dx.$$

We perform the substitution $u = 1 + \frac{9}{4}x$, $\quad du = \frac{9}{4}\,dx$. Then Arclength $= \frac{4}{9}\int_0^2 \sqrt{u} \; du =$

$$\frac{4}{9}\cdot\frac{2}{3}u^{\frac{3}{2}}\bigg|_0^2 = \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{\frac{3}{2}}\bigg|_0^2 = \frac{8}{27}\left(\left(\frac{11}{2}\right)^{\frac{3}{2}} - 1\right) \approx 3.53$$

V. (10) Find the average value of $f(x) = \dfrac{x}{\sqrt{4 - x^2}}$ on the interval $[0, 2]$.

Use a trigonometric substitution to evaluate the integral.

Solution:

We use the substitution $x = 2\sin t$, $\quad dx = 2\cos t\, dt$. Then $f_{ave} = \dfrac{1}{2-0}\displaystyle\int_0^2 \frac{x}{\sqrt{4 - x^2}} \; dx =$

$\frac{1}{2}\int_0^2 \frac{2\sin t}{\sqrt{4-4\sin^2 t}} 2\cos t\ dt = \int_0^2 \frac{\sin t}{2\sqrt{1-\sin^2 t}} 2\cos t\ dt = \int_0^2 \frac{\sin t}{2\sqrt{\cos^2 t}} 2\cos t\ dt =$ $\int_0^2 \frac{\sin t}{2\cos t} 2\cos t\ dt = \int_0^2 \sin t\ dt = \ -\cos t\Big|_0^2$. For the back-substitution, from the trigonometric triangle, we have $\cos t = \frac{\sqrt{4-x^2}}{2}$. Thus, $f_{ave} = -\frac{\sqrt{4-x^2}}{2}\Big|_0^2 = 0-(-1) = 1$.

VI. (10) Consider a thin rod 1 meter long. The mass density of the rod x meters from the left end equals $\rho(x) = \sqrt{1+x^2}$ kg/m, find the center of mass if the rod. (You may use the table of integrals where appropriate.)

Solution:

$$\bar{x} = \frac{\int_0^1 x\rho(x)\ dx}{\int_0^1 \rho(x)\ dx} = \frac{\int_0^1 x\sqrt{1+x^2}\ dx}{\int_0^1 \sqrt{1+x^2}\ dx}$$

We use the u-substitution $u = 1+x^2$, $du = 2x\,dx$ for the integral in the numerator and the table entry # 3 for the integral in the denominator. Thus $\int_0^1 x\sqrt{1+x^2}\ dx = \frac{1}{2}\int_0^1 \sqrt{u}\ du =$ $\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{3}(1+x^2)^{3/2} = \frac{1}{3}(2\sqrt{2}-1)$.

Also, $\int_0^1 \sqrt{1+x^2}\ dx = \left(\frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln|u+\sqrt{x^2+1}|\right)\Big|_0^1 = \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})$.

Thus

$$\bar{x} = \frac{\frac{1}{3}(2\sqrt{2}-1)}{\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})} \approx 0.53.$$

.

VII. Consider the differential equation $y' = e^y\cos x$.

A. (10) Verify that every member of the family of functions

$$y = -\ln(C-\sin x), \qquad C > 1$$

is a solution for the differential equation above. **Show all steps of your argument.**

Solution:

LHS= $(-\ln(C-\sin x))' = -\frac{1}{C-\sin x}(-\cos x) = \frac{\cos x}{C-\sin x}$.

RHS= $e^y\cos x = e^{-\ln(C-\sin x)}\cos x = \frac{1}{e^{\ln(C-\sin x)}}\cos x - \frac{1}{C-\sin x}\cos x = \frac{\cos x}{C-\sin x}$.

Since LHS=RHS, $y = -\ln(C-\sin x)$ is a solution to the differential equation for any constant $C > 1$.

B. (5) Find the solution to differential equation above which also satisfies $y(0) = -1$.

Solution: Substitute $x = 0$ and $y = -1$ in the generals solution $y = -\ln(C - \sin x)$.

We have $-1 = -\ln(C - \sin 0)$. This gives, $1 = \ln C$, or $C = e$. Thus, the solution to the given initial value problem is $y = -\ln(e - \sin x)$.

VIII. (10) Match the following differential equations with the slope field (direction field) below.

Solution:

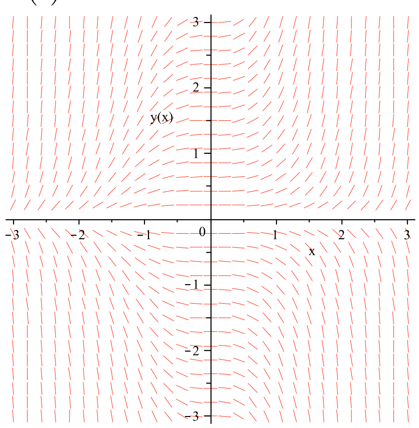
(1) $y' = x^2 + y^2$ has slope field (III) (all slopes are positive or 0 (at the origin))

(2) $y' = x^2 y$ has slope field (I) (slopes are positive above the x-axis and negative below)

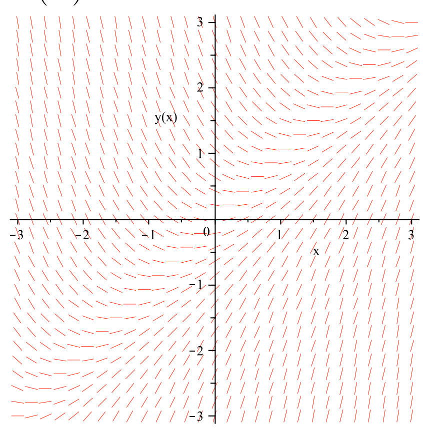
(3) $y' = y + 2$ has slope field (V) (slopes are 0 along the horizontal line $y = 2$)

(4) $y' = xy^2$ has slope field (IV) (slopes are positive to the right of the y-axis and negative to the left)

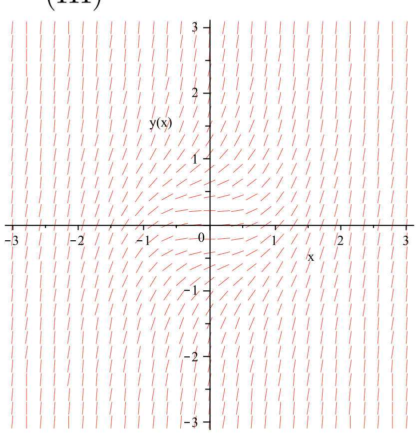
(I)



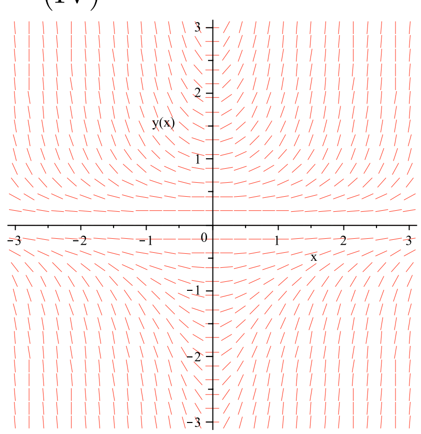
(II)



(III)



(IV)



(V)