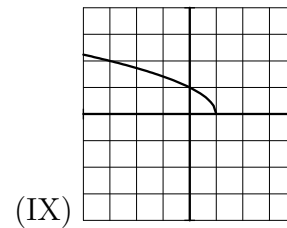
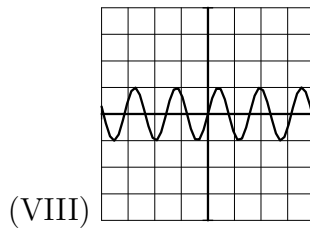
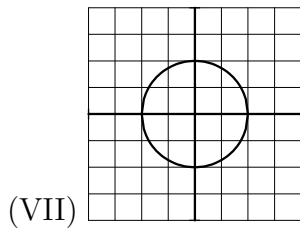
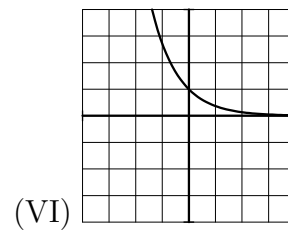
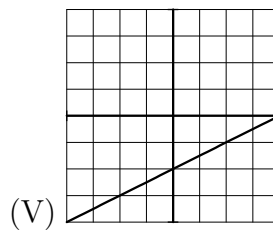
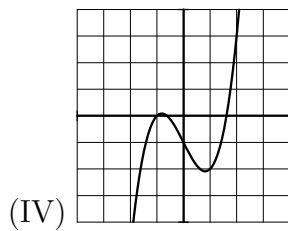
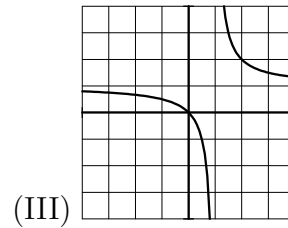
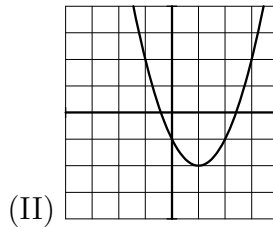
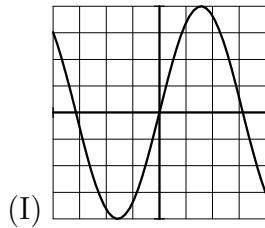


College of the Holy Cross, Fall Semester, 2005
Math 131, Practice Final Answers

1. Circle the number corresponding to the graph of each function. Each square in the figures is 1 unit by 1 unit, and the bold lines are the axes.

- | | |
|------------------------|--|
| (a) $\frac{1}{2}x - 2$ | I II III IV V VI VII VIII IX |
| (b) $\sin(4x)$ | I II III IV V VI VII VIII IX |
| (c) $x^3 - 2x - 1$ | I II III IV V VI VII VIII IX |
| (d) $\frac{x}{x-1}$ | I II III IV V VI VII VIII IX |
| (e) e^{-x} | I II III IV V VI VII VIII IX |
| (f) $x^2 - 2x - 1$ | I II III IV V VI VII VIII IX |
| (g) $4\sin(x)$ | I II III IV V VI VII VIII IX |



2. Compute the following limits.

(a) $\lim_{x \rightarrow 0} \frac{x^2 - 9}{3 - x}$

Answer. Since the numerator and denominator are continuous, and the denominator is nonzero at $x = 0$,

$$\lim_{x \rightarrow 0} \frac{x^2 - 9}{3 - x} = \frac{-9}{3} = -3.$$

(b) $\lim_{x \rightarrow 1} \frac{2^x - 2}{\ln(x)}$,

Answer. The numerator and denominator both approach zero, so by L'Hopital's Rule,

$$\lim_{x \rightarrow 1} \frac{2^x - 2}{\ln(x)} = \lim_{x \rightarrow 1} \frac{2^x \ln 2}{1/x} = 2 \ln 2.$$

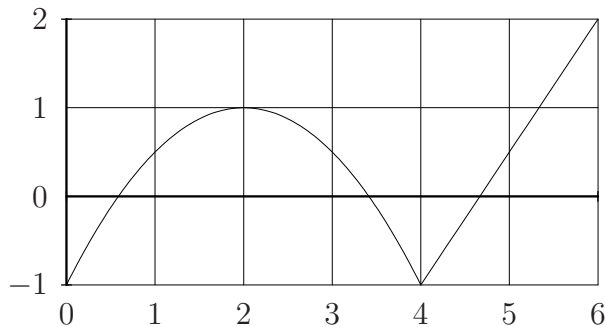
(c) $\lim_{x \rightarrow \infty} \frac{4x^2 + 3e^{-x}}{5x^2 + 7e^{-x}}$

Answer. The numerator and denominator both approach infinity, so by applying L'Hopital's Rule (twice), we get

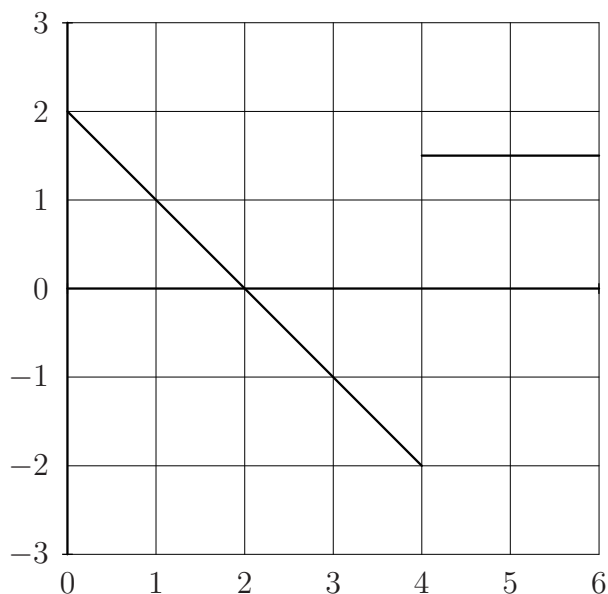
$$\lim_{x \rightarrow \infty} \frac{4x^2 + 3e^{-x}}{5x^2 + 7e^{-x}} = \lim_{x \rightarrow \infty} \frac{8x - 3e^{-x}}{10x - 7e^{-x}} = \lim_{x \rightarrow \infty} \frac{8 + 3e^{-x}}{10 + 7e^{-x}} = \frac{4}{5}$$

3. The graph of the function $f(x)$ is shown below. On the axes provided, sketch the graph of $f'(x)$. Be sure to label the points where f' is zero or undefined.

Graph of f :



Graph of f' :



4. (a) Suppose $f(2) = 3$ and $f'(2) = -5$. Use the linear approximation of f at $x = 2$ to estimate the value of $f(1.97)$.

Answer. The linear approximation of f at $x = 2$ is $f(x) \approx f(2) + f'(2)(x - 2) = 3 - 5(x - 2)$, so $f(1.97) \approx 3 - 5(1.97 - 2) = 3 - 5(-0.03) = 3.15$.

- (b) Suppose as in part (a) that $f(2) = 3$ and $f'(2) = -5$. Also suppose $g(3) = -1$ and $g'(3) = 7$. Let $h(x) = g(f(x))$. Find $h'(2)$.

Answer. By the chain rule, $h'(x) = g'(f(x)) \cdot f'(x)$. At $x = 2$, $h'(2) = g'(f(2)) \cdot f'(2) = g'(3) \cdot (-5) = 7(-5) = -35$.

5. Compute the derivative of each function.

(a) $f(x) = \frac{x^3}{2x^2 + 1}$

Answer. By the quotient rule,

$$f'(x) = \frac{(2x^2 + 1)(3x^2) - x^3(4x)}{(2x^2 + 1)^2} = \frac{x^2(2x^2 + 3)}{(2x^2 + 1)^2}.$$

(b) $g(x) = x^2 \cos(3x)$

Answer. By the product and chain rules,

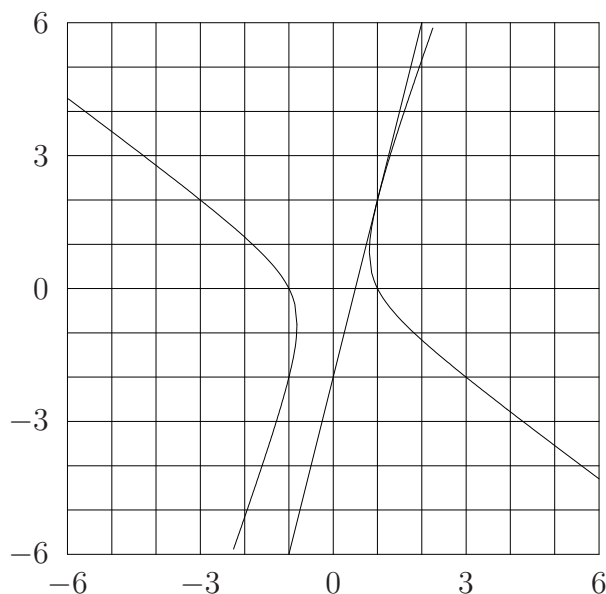
$$g'(x) = -3x^2 \sin(3x) + 2x \cos(3x) = x(-3x \sin(3x) + 2 \cos(3x)).$$

(c) $h(x) = \ln(1 + e^{\sqrt{x}})$

Answer. By the chain rule,

$$h'(x) = \frac{1}{1 + e^{\sqrt{x}}} \cdot e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}(1 + e^{\sqrt{x}})}.$$

6. The hyperbola $2x^2 + 2xy - y^2 = 2$ is shown below.



- (a) Compute $\frac{dy}{dx}$ in terms of x and y .

Answer. Differentiating with respect to x gives

$$4x + 2x \frac{dy}{dx} + 2y - 2y \frac{dy}{dx} = 0 \quad \implies \quad \frac{dy}{dx} = \frac{2x + y}{y - x}$$

- (b) Find the equation of the tangent line to the curve at the point $(1, 2)$ and sketch the tangent line on the figure above.

Answer. At $(1, 2)$, $\frac{dy}{dx} = 4$, so the equation of the tangent line is $y - 2 = 4(x - 1)$, or $y = 4x - 2$.

7. Suppose the position of an object is described by the parametric curve $x = t^3 - t^2$, $y = 3t^3 - 4t$, where t is measured in seconds.

- (a) Find the location of the object at time 2 seconds. Find the equation of the tangent line to the curve at this point.

Answer. The object is located at $x = 4$, $y = 16$. The tangent line has slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2 - 4}{3t^2 - 2t} = \frac{32}{8} = 4$$

when $t = 2$. Therefore the equation of the tangent line is $y - 16 = 4(x - 4)$, or $y = 4x$.

- (b) At what time is the speed of the object zero?

Answer. The speed is zero when both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero. Since $\frac{dx}{dt} = 3t^2 - 2t = 0$ when $t = 0$ and $t = -2/3$, and $\frac{dy}{dt} = 9t^2 - 4 = 0$ when $t = \pm 2/3$, the speed is zero only when $t = \frac{2}{3}$.

8. Let $f(x) = x^3 e^x$.

- (a) Find and classify (local min/max, or neither) the critical points of f .

Answer. By the product rule, $f'(x) = 3x^2e^x + x^3e^x = e^xx^2(3+x)$, so the critical points of f are $x = 0$ and $x = -3$. Since $f'(x) < 0$ for $x < -3$, $f'(x) > 0$ for $-3 < x < 0$ and $f'(x) > 0$ for $x > 0$, the first derivative test implies that f has a local minimum at $x = -3$, but neither a local max nor a local min at $x = 0$.

- (b) Find the inflection points of f .

Answer. By the product rule again, $f''(x) = e^x(6x + 3x^2) + e^x(3x^2 + x^3) = xe^x(6 + 6x + x^2)$. The roots of f'' are therefore $x = 0$ and $x = -3 \pm \sqrt{3}$. These are all inflection points since f'' changes sign at each of them.

9. You have \$3 with which to construct a box with a square base. The material for the top and bottom costs 2 cents per square inch, while the material for the sides costs 1 cent per square inch. What dimensions maximize the volume of the box? *Hint:* First write the cost of the box and the volume of the box in terms of the dimensions of the box.

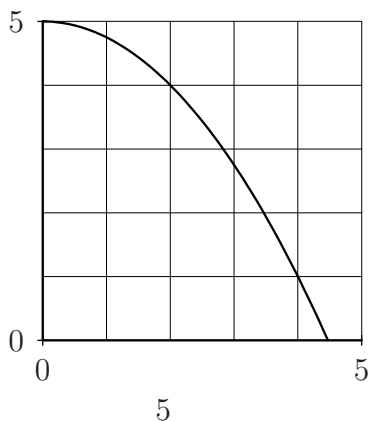
Answer. Let x denote the side length of the square, and y the height. Then the volume is $V = x^2y$ and the cost of the box is $C = 2(2x^2) + 1(4xy)$. Since we have \$3 to use, this means $4x^2 + 4xy = 300$, and solving for y gives $y = (300 - 4x^2)/(4x)$. Therefore the volume is $V(x) = \frac{1}{4}x(300 - 4x^2) = \frac{1}{4}(300x - 4x^3)$. The domain is $0 \leq x \leq \sqrt{300/4}$. Solving $V'(x) = \frac{1}{4}(300 - 12x^2) = 0$ gives the critical point $x = 5$ ($x = -5$ is not in the domain). Since $V(0) = V(\sqrt{300/4}) = 0$, the maximum volume is $V(5) = 250$, and the dimensions that produce this volume are $x = 5$ and $y = 10$.

10. Suppose you are making a snow-person, and you begin by rolling a snowball in the snow. Assume the snowball always remains in the shape of a sphere. If the radius of the snowball increases at a constant rate of 2 inches per second, how fast is the volume of the snowball increasing when the radius is 6 inches? Give the units of your answer. The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Answer. We are given that $\frac{dr}{dt} = 2$ and want to find $\frac{dV}{dt}$ when $r = 6$. Differentiating the volume formula gives

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4\pi(6)^2(2) = 288\pi \text{ in}^3/\text{sec}$$

11. The graph of $f(x) = 5 - \frac{1}{4}x^2$ is shown below.



- (a) Compute the left and right hand sums for f over the interval $1 \leq x \leq 4$ using $n = 3$ subintervals. On the figure above sketch the rectangles which represent the terms in the left and right hand sums.

Answer. $\Delta x = 1$ so

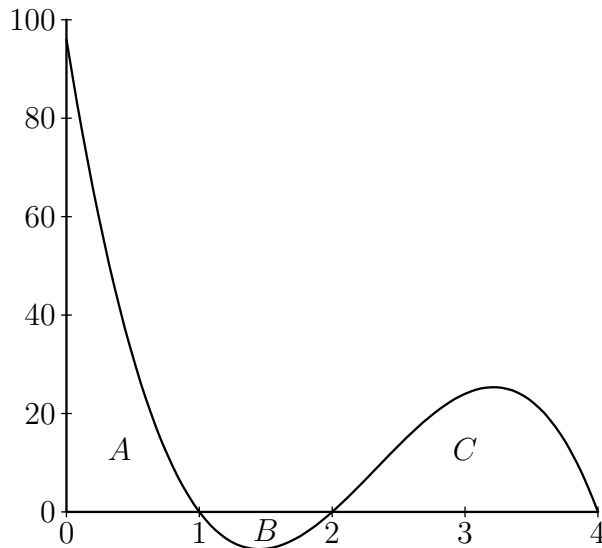
$$\text{LHS} = f(1) + f(2) + f(3) = (5 - \frac{1}{4}) + (5 - 1) + (5 - \frac{9}{4}) = 11.5$$

$$\text{RHS} = f(2) + f(3) + f(4) = (5 - 1) + (5 - \frac{9}{4}) + (5 - 4) = 7.75$$

- (b) Which sum is a better estimate of the exact area under the graph of f over $1 \leq x \leq 4$? Explain.

Answer. Because the graph of f is decreasing and concave down, the left hand sum gives a better estimate.

12. The graph of $f(x)$ is shown below. The areas of the regions shown are $A = 37$, $B = 5$ and $C = 32$.



- (a) Evaluate $\int_0^4 f(x) dx$

Answer. $\int_0^4 f(x) dx = A - B + C = 64.$

- (b) Find the average value of f over the interval $1 \leq x \leq 4$.

Answer. The average value is

$$\frac{1}{4-1} \int_1^4 f(x) dx = \frac{1}{3} (-B + C) = \frac{1}{3} (-5 + 32) = 9.$$

13. Suppose the rate of growth of a population of bats living under a bridge, in thousands of bats per year, is

$$f(t) = 2 + \frac{t}{3}$$

where t is measured in years since 1990. Suppose there were 40 thousand bats in 1990.

- (a) Write out an integral that represents the change in the bat population from 1990 to 2005.

Answer. Let $F(t)$ be the population (in thousands) in year t . Then $F'(t) = f(t)$, so by the Fundamental Theorem of Calculus,

$$F(15) - F(0) = \int_0^{15} 2 + \frac{1}{3}t \, dt$$

is the change in the bat population from 1990 to 2005.

- (b) Compute the integral exactly (use the graph of f) and determine what the population will be in 2005.

Answer. Since f is a linear function, the region under its graph is the union of a rectangle and a triangle. The total area under the graph is $15(2) + \frac{1}{2}(15)(5) = 67.5$, so

$$\int_0^{15} 2 + \frac{1}{3}t \, dt = 67.5.$$

Solving for $F(15)$ gives $F(15) = F(0) + 67.5 = 40 + 67.5 = 107.5$, which means there are 107,500 bats in 2005.