Inverse functions and logarithms

Given a one-to-one function f with domain A and range B, the inverse function of f, denoted f^{-1}, has domain B and range A and is defined by

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x.$$

In other words, the function f^{-1} maps x to the number which f would map back to x.

Note: The functions f and f^{-1} undo each other. This means that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

The exponential function with base a, $f(x) = a^x$, is the inverse function of the logarithmic function with base a, $g(x) = \log_a x$ (and vice-versa, the logarithmic function is the inverse of the exponential function). Thus,

$$\log_a(a^x) = x, \quad a^{\log_a x} = x$$

For natural logarithms this means.

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

We also have

$$\ln e = 1 \quad \text{and} \quad \log_a a = 1.$$

Laws of exponents: $a^{x+y} = a^x \cdot a^y$, $a^{x-y} = \frac{a^x}{a^y}$, $(a^x)^y = a^{xy}$.

Examples: $3^5 = 3^{2+3} = 3^2 \cdot 3^3 = 9 \cdot 27 = 243$, $\frac{3^7}{3^2} = 3^{7-2} = 3^5 = 243$, $(3^2)^3 = 3^6 = 729$

Laws of logarithms: $\log_a(x \cdot y) = \log_a x + \log_a y$, $\log_a(\frac{x}{y}) = \log_a x - \log_a y$, $\log_a(x^r) = r \log_a x$.

Example: $\ln 3 + \ln 13 - 3 \ln 4 = \ln(3 \cdot 13) - \ln 4^3 = \ln \frac{39}{4^3}$.

Exercises: Find a formula for the inverse of the function.

1) $f(x) = \dfrac{1+3x}{5-2x}$ 2) $f(x) = \sqrt{2+5x}$ 3) $f(x) = 5 - 4x^3$ 4) $f(x) = 2^{10^x}$

5) Find the domain and the range of the function $g(x) = \ln(4 - x^2)$.

Solve each equation for x.

6) $e^x = 16$, 7) $\ln x = -1$, 8) $\ln(2x - 1) = 3$, 9) $e^{3x-4} = 2$

10) $10^{x+3} = 5e^{7-x}$

11) $2x - 1 = e^{\ln x^2}$

Simplify the expressions (A and B are positive constants):

12) $e^{\ln(1/2)}$

13) $10^{\log(AB)}$

14) $5e^{\ln(A^2)}$

15) $\ln(e^{2AB})$

16) $\ln(1/e) + \ln(AB)$

17) $2\ln(e^A) + 3\ln(B^e)$

Answers:

1) $f^{-1}(x) = \dfrac{5x-2}{2x+3}$

2) $f^{-1}(x) = \dfrac{x^2-2}{5}$

3) $f^{-1}(x) = \left(\dfrac{5-x}{4}\right)^{1/3}$

4) $f^{-1}(x) = \log_{10}\log_2 x$

5) $(-2, 2)$, $(-\infty, \ln 4]$

6) $4\ln 2$

7) $1/e$

8) $\frac{1}{2}(e^3+1)$

9) $\frac{1}{3}(\ln 2 + 4)$

10) $x = \dfrac{\ln 5 - \ln 1000 + 7}{\ln 10 + 1} \approx 0.515$

11) $x = 1$

12) $\frac{1}{2}$

13) AB

14) $5A^2$

15) $2AB$

16) $-1 + \ln A + \ln B$

17) $2A + 3e\ln B$