College of the Holy Cross, Spring 2009
Math 241 - Chapter 1 practice problems

1. Consider the points A, B, C of coordinates $(1, 2, 1)$, $(-1, 0, 1)$ and $(2, 1, 0)$ respectively. Find the area of the triangle ABC.

2. Are the points $A(1, 0, 1)$, $B(1, 3, 4)$, $C(0, 2, 2)$ and $D(3, 2, 5)$ in the same plane? Justify your answer.

3. Consider the points A, B, C of coordinates $(1, 2, 1)$, $(-1, 0, 1)$ and $(2, 1, 0)$ respectively. Find the coordinates of a point D such that $ABCD$ is a parallelogram with diagonal BD.

4. Find the angle between the diagonal of a cube and one of the diagonals of its faces, starting from the same vertex.

5. Find an equation for the plane passing through the origin which is parallel to the lines

$$L_1 : (t, 1 - t, 1 + t) \quad \text{and} \quad L_2 : (2, 2t + 1, -t)$$

6. In cylindrical coordinates, the equation $r = 2$ describes:

   A. A plane B. A paraboloid C. A cone D. A sphere E. A cylinder

7. In spherical coordinates, the equation $\rho = \cos\varphi$ describes (transform the equation in rectangular coordinates before answering):

   A. A plane B. A paraboloid C. A cone D. A sphere E. A cylinder

8. Sketch and describe in words the following surfaces:

   (a) $x^2 + y^2 + z^2 = 2x + 2y + 2z$
   (b) $x^2 - 2y^2 - 4z^2 = 0$
   (c) $x^2 - 2y^2 - 4z^2 = 5$
   (d) $x - 2y^2 - 4z^2 = 1$

9. Find the distance from the point $A(1, 0, -1)$ to the line of intersection of the two planes: $x - y + z = 3$ and $x + 2y - z = 0$.

10. **Textbook exercises:** Sec. 1.8: 17, 19, 23, 25, 28

Solutions.

1. The area of the triangle is:

$$Area(ABC) = |\overline{AB} \times \overline{AC}|/2.$$

Since $\overline{AB} \times \overline{AC} = (-2,-2,0) \times (1,-1,-1) = (2,-2,4)$, it follows $Area(ABC) = \sqrt{24}/2 = \sqrt{6}$.

2. There are a few ways of doing this problem: we can find the equation of the plane containing A, B, C and check whether the coordinates of D satisfy it; compute two normal vectors $AB \times AC$ and $AB \times AD$ and check whether they are parallel; or use the box product $AB \cdot (AC \times AD)$, which is 0 exactly when the four points are coplanar.

Using the first method, the plane of the triangle ABC has normal vector

$$AB \times AC = (-3,-3,3) = 3(-1,-1,1)$$

hence the equation of the plane going through A, B, C is (taking A as a point on the plane)

$$-(x-1) - (y-0) + z - 1 = 0$$

or $x + y - z = 0$. Since the coordinates of D satisfy $3 + 2 - 5 = 0$, D is also on that plane, so the four points are in the same plane.

3. If the coordinates of D are (x, y, z), the fact that the midpoints of the two diagonals AC and BD in the parallelogram coincide implies:

$$\frac{(1,2,1) + (2,1,0)}{2} = \frac{(-1,0,1) + (x,y,z)}{2}$$

so $x = 4, y = 3, z = 0$. That is D has coordinates $(4, 3, 0)$.

4. Take the cube to be the one formed by the unit vectors i, j, k. Then a diagonal of the cube is $i + j + k$ and the diagonal of a face starting in the same vertex is $i + j$. From the properties of the dot product:

$$(i + j + k) \cdot (i + j) = 2 = \sqrt{3}\sqrt{2}\cos\theta,$$

hence $\cos\theta = 2/\sqrt{6} = \sqrt{6}/3$. Therefore $\theta = \cos^{-1}\sqrt{6}/3$.

5. The normal of the plane is perpendicular to the direction vectors of the two lines, which are obtained by looking at the coordinates of t: $(1,-1,1)$ and $(0,2,-1)$. Therefore the normal vector is:

$$\mathbf{n} = (1,-1,1) \times (0,2,-1) = (-1,1,2)$$

and since the plane passes through $(0,0,0)$, its equation is: $-x + y + 2z = 0$.

6. The answer is E. In rectangular coordinates, $r = \sqrt{x^2 + y^2}$, so $x^2 + y^2 = 4$ which is a cylinder centered around the z-axis.

7. The answer is D. The equation can also be written $\rho^2 = \rho\cos\varphi$, which in rectangular coordinates becomes:

$$x^2 + y^2 + z^2 = z, \quad \text{or} \quad x^2 + y^2 + (z - 1/2)^2 = 1/4,$$

which is a sphere centered at $(0, 0, 1/2)$ of radius $1/2$.

8. (a) Completing the square: $(x-1)^2 + (y-1)^2 + (z-1)^2 = 3$, which is a sphere of radius $\sqrt{3}$, centered at $(1, 1, 1)$.

(b) An elliptic cone with opening around the x-axis. The sections in planes x =constant, are ellipses, while for $z = 0$, $x = \pm\sqrt{2}y$ are two lines.

(c) A hyperboloid of two sheets, around the x-axis. The sections in planes x =constant, are ellipses, if $x^2 - 5 = 2y^2 + 4z^2 \le 0$, that is $|x| \le \sqrt{5}$. If $z = 0$, $x^2 - 2y^2 = 5$ is a hyperbola opening along the x-axis.

(d) Paraboloid, opening in the positive x direction, with the tip at $(1, 0, 0)$.

9. First we find two points on the line of intersection. Taking $z = 0$ and solving for x and y in the system:

$$x - y = 3, x + 2y = 0$$

we find $x = 2$, $y = -1$. Therefore the point B of coordinates $(2, -1, 0)$ is on both planes. Similarly take $z = 3$ and solve for:

$$x - y = 0, x + 2y = 3$$

to find the point $C(1, 1, 3)$. Now that we have two points B, C on the line we can find the distance from A to it:

$$d = \frac{||\mathbf{BC} \times \mathbf{AB}||}{||\mathbf{BC}||}$$

where $\mathbf{BC} = (-1, 2, 3)$, $\mathbf{AB} = (1, -1, 1)$. The computation gives $d = \sqrt{3}$.