MAT 136, Fall 2008 - First Midterm Solutions
Friday, September 26

1. (20 points) Mark as true or false. You do not have to justify your answers for this problem.

(a) The function $y = e^x$ has no horizontal asymptotes.	True◯	False◯
(b) The function $y = \ln(1 + x^2)$ is one-to-one.	True◯	False◯
(c) The function $f(x) = \dfrac{\lvert x\rvert}{x}$ is continuous on the interval $[-1, 1]$.	True◯	False◯
(d) The function $f(x) = \lvert x\rvert$ is continuous on the interval $[-1, 1]$.	True◯	False◯
(e) The functions $f(x) = x^3 + 1$ is odd.	True◯	False◯

Solution. a) False. e^x has an asymptote as $x \to -\infty$.

b) False. The function takes the same value at x and $-x$, so it is not one-to-one.

c) False. The function is 1 if $x > 0$ and -1 if $x < 0$, so the left and right limits are not the same.

d) True.

e) False. $f(-1) = 0$, $f(1) = 2$, so the function is not odd.

2. (20 points) a) Find all the vertical and horizontal asymptotes of the function $f(x) = \dfrac{1 + e^x}{1 - e^x}$.

Solution. The denominator approaches 0 when $x \to 0$, so the function has a vertical asymptote at $x = 0$.

It also has 2 horizontal asymptotes:

$$\lim_{x\to\infty} \frac{1 + e^x}{1 - e^x} = \lim_{x\to\infty} \frac{1/e^x + 1}{1/e^x - 1} = \frac{0 + 1}{0 - 1} = -1$$

(after dividing by the fastest growing term to compute the limit). Also

$$\lim_{x\to-\infty} \frac{1 + e^x}{1 - e^x} = \frac{1 + 0}{1 - 0} = 1$$

because $\lim_{x\to-\infty} e^x = 0$. So $y = 1$ and $y = -1$ are horizontal asymptotes

b) Compute the limit: $\lim_{x\to1} \dfrac{\sqrt{x} - 1}{x - 1}$.

Solution. Multiply by conjugate:

$$\lim_{x\to1} \frac{\sqrt{x} - 1}{x - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} = \lim_{x\to1} \frac{x - 1}{(x - 1)(\sqrt{x} + 1)} = \lim_{x\to1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}.$$

c) Compute the limit: $\lim_{x\to\infty} \dfrac{x^2 + 1}{\sqrt{x^4 + 1}}$.

Solution. Both the numerator and the denominator have fastest growing term $x^2 = \sqrt{x^4}$, so we divide by it:

$$\lim_{x\to\infty} \frac{x^2 + 1}{\sqrt{x^4 + 1}} = \lim_{x\to\infty} \frac{1 + 1/x^2}{\sqrt{1 + 1/x^4}} = 1$$

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3. (20 points) The bee population in an apiary is 4 thousands at time $t = 0$. The population at time t is given by the function $P(t) = \dfrac{100}{1+24e^{-t}}$, where time is measured in years and the population is measured in thousands.

a) What is $\lim_{t\to\infty} P(t)$?

b) Compute the inverse function P^{-1}, and explain its meaning.

c) After how many years does the bee population reach 80 thousands? Leave your answer as a logarithm.

Solution. a) Since $\lim_{t\to\infty} e^{-t} = 0$, the limit equals $100/1 = 100$.

b) Solving $y = \dfrac{100}{1+24e^{-t}}$ for t gives:

$$t = -\ln\frac{100/y-1}{24},$$

so the inverse function is $P^{-1}(y) = -\ln\frac{100/y-1}{24}$. It represents the time when the population reaches y thousands.

c) From part b), $P^{-1}(80) = -\ln\frac{100/80-1}{24} = -\ln(1/96) = \ln 96$ years.

4. (15 points) a) Find the cartesian equation of the curve whose parametric equations are $x = \sin(t)$, $y = 2\cos(t)$, for $0 \le t \le \pi$.

b) Sketch the curve in part a), for the given range of t.

Solution. a) Using the identity $\sin^2(t) + \cos^2(t) = 1$ and the parametric equations gives:

$$x^2 + y^2/4 = 1,$$

which is the equation of an ellipse.

b) Plugging in $t = 0, \pi/2, \pi$ we find the points $(0,2), (1,0), (0,-2)$, so when t varies between 0 and π, we get the right half of the ellipse (from the point (0,2) to (0,-2) traced clockwise).

5. (25 points) a) Find the power function $f(x) = Cx^a$ whose graph passes through the points $(2,3)$, $(4,15)$.

Solution. The two points give two equations, $f(2) = 3$ and $f(4) = 15$:

$$3 = C2^a, \quad 15 = C4^a.$$

From the first $C = 3/2^a$, and plugging into the second gives $15 = 3\cdot 4^a/2^a$, that is $5 = 2^a$. So $a = \log_2 5$, $C = 3/2^{\log_2 5} = 3/5$, and the function is $f(x) = \frac{3}{5}x^{\log_2 5}$.

b) Find the inverse function of $f(x) = \ln(1+2x)$. What is the range of the inverse function f^{-1}?

Solution. Solving $y = \ln(1+2x)$ for x gives $x = \frac{e^y-1}{2}$, so the inverse function is

$$f^{-1}(x) = \frac{e^x-1}{2}.$$

The range of f^{-1} is the domain of $f(x) = \ln(1+2x)$, that is $1+2x > 0, x > -1/2$.

c) Give an example of two functions f and g with $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to 0} g(x) = \infty$, and $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 3$.

Solution. Take $g = 1/x^2$ and $f = 3/x^2$.

d) Do the graphs of the functions $y = e^x$ and $y = 3x$ intersect? Justify your answer using the Intermediate value theorem. Note: $e = 2.71...$

Solution. Let $f(x) = e^x - 3x$. Then $f(0) = 1 > 0$ and $f(1) = e - 3 < 0$. By the Intermediate value theorem there is a point c between 0 and 1 with $f(c) = 0$, and at that point the graphs intersect: $e^c = 3c$.