MAT 132, Spring 2009 - First Midterm Solutions
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Formulas that may be useful:

$$\cos^2(t) = \frac{1+\cos(2t)}{2}, \quad \sin^2(t) = \frac{1-\cos(2t)}{2}, \quad \sin(2t) = 2\sin(t)\cos(t)$$

$$\int \sqrt{u^2-a^2}du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2-a^2}| + C$$

$$\int \sqrt{u^2+a^2}du = \frac{u}{2}\sqrt{u^2+a^2} - \frac{a^2}{2}\ln|u+\sqrt{u^2+a^2}| + C$$

$$\int \sqrt{a^2-u^2}du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int \frac{\sqrt{u^2-a^2}}{u}du = \sqrt{u^2-a^2} - a\cos^{-1}(a/|u|) + C$$

$$\int \sec(\theta)d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

1. (15 points) a) Find the area under the parabola $y = 4x - x^2$ and above the x axis.

Solution. The x intercepts of the parabola are where $y = 0$: $4x - x^2 = 0$, that is $x = 0$ and $x = 4$. Therefore the area equals:

$$\int_0^4 4x - x^2 dx = 2x^2 - \frac{x^3}{3}\Big|_0^3 = 32/3.$$

b) Compute the derivative of the following function: $F(x) = \int_x^0 \ln(1+t^2)\ dt$

Solution. $F(x) = -\int_0^x \ln(1+t^2)\ dt$, so the Fundamental Theorem of calculus tells us that

$$F'(x) = -\ln(1+x^2).$$

c) Compute the derivative of the following function: $G(x) = \int_0^{\sqrt{x}} e^{t^2}\ dt$

Solution. If $H(x) = \int_0^x e^{t^2}\ dt$, then the Fundamental Theorem of calculus tells us that $H'(x) = e^{x^2}$. The given function is $G(x) = H(\sqrt{x})$. To compute its derivative we apply the chain rule:

$$G'(x) = H'(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{e^x}{2\sqrt{x}}$$

2. (15 points) a) Integrate using an appropriate u-substitution: $\int_1^e \frac{(\ln x)^2}{x}\ dx$

Solution. Substitute $u = \ln(x)$, $du = \frac{1}{x}\ dx$. Note the limits of integration change to $\ln(1) = 0$ to $\ln(e) = 1$:

$$\int_0^1 u^2 du = \frac{u^3}{3}\Big|_0^1 = 1/3.$$

b) Integrate using an appropriate u-substitution: $\int \sin^3(x)\cos^9(x)\ dx$

Solution. Write the integral as :

$$\int \sin^2(x)\cos^9(x)\sin(x)\ dx$$

and use the substitution $u = \cos(x)$, $du = -\sin(x)dx$ (the substitution $u = \sin x$ would also work, after separating out a $\cos(x)$). Since $\sin^2(x) = 1 - \cos^2(x)$, the integral becomes:

$$\int (1-u^2)u^9\ (-du) = \int u^{11} - u^9\ du = u^{12}/12 - u^{10}/10 = \cos^{12}(x)/12 - \cos^{10}(x)/10 + C$$

c) Integrate using integration by parts: $\int x\sin(2x)\ dx$.

Solution. $u = x \qquad du = dx$

$dv = \sin(2x)\,dx \quad v = -\cos(2x)/2$:

$$\int x\sin(2x)\ dx = -x\cos(2x)/2 - \int -\frac{\cos(2x)}{2}\ dx = -\frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4} + C.$$

3. (30 points) Compute the following integrals. You may need to refer to the list of integrals on the front page.

a) $\int \frac{x}{x^2+x-2}\ dx$

Solution. Use partial fraction decomposition. Since $x^2+x-2 = (x-1)(x+2)$:

$$\frac{x}{x^2+x-2} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Multiply by the common denominator: $x = A(x+2) + B(x-1)$. Now plug in $x = 1$ to get $A = 1/3$ and $x = -2$ to get $B = 2/3$. Finally:

$$\int \frac{x}{x^2+x-2}\ dx = \int \frac{1/3}{x-1} + \frac{2/3}{x+2}dx = \frac{1}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + C$$

b) $\int x\sqrt{x^4+1}\ dx$

Solution. Substitute $u = x^2$, $du = 2x\ dx$:

$$\int x\sqrt{x^4+1}\ dx = \int \sqrt{u^2+1}\ \frac{du}{2} = \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2+1} - \frac{1}{2}\ln|u + \sqrt{u^2+1}|\right) + C$$

using the formula on the front page with $a = 1$. Plugging back $u = x^2$. gives the answer.

c) $\int \sqrt{e^{2x}-1}\ dx$. Hint: $e^{2x} = (e^x)^2$.

Solution. Substitute $u = e^x$, $du = e^x\ dx$. We also need to solve for dx: $dx = du/e^x = du/u$:

$$\int \sqrt{e^{2x}-1}\ dx = \int \sqrt{u^2-1}\frac{du}{u} = \sqrt{u^2-1} - \cos^{-1}(1/|u|) + C$$

where we used the formula on the front page. Plugging back $u = e^x$ gives the answer.

4. (10 points) Use a trigonometric substitution to compute the integral:

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx.$$

You may use the list of integrals on the first page, after making the trigonometric substitution.

Solution. Since $1+\tan^2\theta = \sec^2\theta$, the substitution that gets rid of the square root is $x = \tan\theta$, so $\sqrt{1+x^2} = \sec\theta$. For this substitution, $dx = \sec^2\theta\ d\theta$:

$$\int \frac{1}{\sqrt{1+x^2}}\ dx = \int \frac{1}{\sec\theta}\sec^2\theta d\theta = \int \sec\theta d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

using the formula on the first page. Before plugging in the limits, we need to get the answer in terms of x: $\tan(\theta) = x, \quad \sec\theta = \sqrt{1+x^2}$.

$$\int_0^1 \frac{1}{\sqrt{1+x^2}}\ dx = \ln|\sqrt{1+x^2}+x|\Big|_0^1 = \ln(1+\sqrt{2}).$$

5. (10 points) Compute the following improper integrals, or show they are divergent. Justify your answers.

a) $\displaystyle\int_0^1 \frac{1}{x^2}\ dx$

Solution. The integrand is not defined at $x=0$:

$$\int_0^1 \frac{1}{x^2}\ dx = -\frac{1}{x}\Big|_0^1 = -1 + \lim_{x\to 0^+}\frac{1}{x} = \infty$$

so the integral diverges.

b) $\displaystyle\int_1^\infty \frac{\ln x}{x^{3/2}}\ dx$

Solution. Integrate by parts to find the antiderivative:

$u = \ln x \qquad du = \frac{1}{x}\ dx$

$dv = x^{-3/2}\ dx \quad v = -2x^{-1/2} = -\dfrac{2}{\sqrt{x}}$:

$$\int \frac{\ln x}{x^{3/2}}\ dx = -\frac{2\ln x}{\sqrt{x}} - \int -\frac{2}{\sqrt{x}}\frac{1}{x}dx = -\frac{2\ln x}{\sqrt{x}} + 2\int x^{-3/2}dx = -\frac{2\ln x}{\sqrt{x}} + 4/\sqrt{x} + C$$

So

$$\int_1^\infty \frac{\ln x}{x^{3/2}}\ dx = \lim_{x\to\infty} -\frac{2\ln x}{\sqrt{x}} + 4/\sqrt{x} + 4 = 4$$

where we have used that: $\lim_{x\to\infty} 4/\sqrt{x} = 0$ and by L'Hospital's rule:

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty}\frac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$$