Third midterm solutions
MAT 131, Fall 2008

1. (20 points) Compute the derivatives of the following functions. You do not need to simplify your answer.

a) $y = \sqrt{1+x^3}$

Solution. Use the power rule and the chain rule:

$$y' = [(1+x^3)^{1/2}]' = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}$$

.

b)$y = \dfrac{1+\ln(2x)}{\cos(x)}$

Solution. Use the quotient rule and the chain rule for the derivative of $\ln(2x)$:

$$y' = \frac{(2/(2x))\cos(x) - (1+\ln(2x))(-\sin(x))}{\cos^2(x)} = \frac{\cos(x)/x + (1+\ln(2x))\sin(x)}{\cos^2(x)}$$

c) $y = xe^{\sqrt{x}}$

Solution. Use the product rule and the chain rule for the derivative of $e^{\sqrt{x}}$:

$$y' = e^{\sqrt{x}} + xe^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} = e^{\sqrt{x}} + \frac{1}{2}\sqrt{x}e^{\sqrt{x}}.$$

d) $y = x^{\sin x}$

Solution. Use logarithmic differentiation: first apply ln to get $\ln y = \sin(x)\ln(x)$; then differentiate the last equation:

$$\frac{y'}{y} = \cos(x) \cdot \ln(x) + \sin(x) \cdot \frac{1}{x}.$$

Finally solve for y': $y' = x^{\sin x}\left[\cos(x) \cdot \ln(x) + \frac{\sin(x)}{x}\right]$.

2. (10 points) The equation $x^2y^2 + x\ln(y) = 4$ determines y as an implicit function of x.

a) Compute the derivative y'.

b) Find the equation of the tangent line to the graph of y at the point $(2,1)$.

Solution. a) Use implicit differentiation, thinking of y as a function of x:

$$2x^2yy' + 2xy^2 + \ln(y) + x\frac{y'}{y} = 0.$$

Then take y' as a common factor and solve for it:

$$y' = \frac{-xy^2 - \ln(y)}{2x^2y + x/y}.$$

b) When $x = 2, y = 1$, the slope is $y' = -2/10 = -1/5$ (note: $\ln(1) = 0$). Now use the point-slope formula to find the equation of the tangent:

$$y - 1 = (-1/5)(x-2).$$

3. (10 points) a) Find the linear approximation of the function $f(x) = \sqrt[3]{x}$ at the point $x = 8$.

b) Estimate $\sqrt[3]{8.3}$, using linear approximation.

c) Is the approximation in part b) an overestimate or underestimate of the actual value? Justify your answer, without using the calculator to compute the actual value.

Solution. a) The linear approximation of $f(x)$ at $x = 8$ is $L(x) = f(8)+f'(8)(x-8)$. Since $f(x) = x^{1/3}$, its derivative is $f'(x) = (1/3)x^{-2/3}$. So $f(8) = 2, f'(8) = 1/12$ and the linear approximation is:

$$L(x) = 2 + \frac{1}{12}(x - 8).$$

b) $\sqrt[3]{8.3} = f(8.3) \simeq L(8.3) = 2 + \frac{1}{12}0.3 = 2.025$.

c) The linear approximation represents the y-coordinate of the tangent line to $f(x)$ at $x = 8$. Sketching the graphs of $f(x)$ and of the tangent line, we see that the tangent line is above the graph, so the linear approximation is an overestimate. Indeed the actual value is $\sqrt[3]{8.3} = 2.0232$.

4. (10 points) A rocket is launched vertically and is tracked by a ground station 3 miles from the launch pad. What is the vertical speed of the rocket when its height above the ground is 4 miles, and its distance to the ground station is increasing at 3600 miles per hour?

Solution. First draw a picture with the rocket in flight, and label by y the height of the rocket above the ground. Also label by z the distance from the rocket to the ground station. We are given that $dz/dt = 3600$, and we want to find dy/dt when $y = 4$.

Using the Pythagorean theorem, we can relate y and z:

$$3^2 + y^2 = z^2.$$

Note that the distance between the ground station and the launch pad does not change, it is equal to 3. Taking derivative, we have $2y\frac{dy}{dt} = 2z\frac{dz}{dt}$, so $\frac{dy}{dt} = \frac{z}{x}\frac{dz}{dt}$. Now we can plug in $x = 4$, and find that $z = 5$ from Pythagora's theorem, so $\frac{dy}{dt} = 4500$.

5. (20 points) Consider the function $f(x) = x^3 - 12x + 2$.

a) Find the critical numbers of f and determine on which interval f is increasing/decreasing.

b) Find the local minima and maxima of f. Justify your answer.

c) Find the intervals on which f is concave up/concave down and determine all points of inflection.

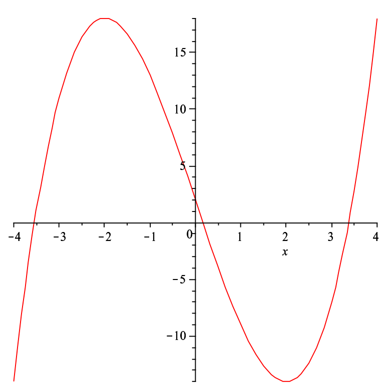
d) Sketch the graph of f, clearly marking the local extrema and inflection points.

Solution. a) $f'(x) = 3x^2 - 12 = 3(x^2 - 4)$. The critical numbers are where $f'(x) = 0$, that is $x = 2$ and $x = -2$. The function is increasing when $f' > 0$, and plugging in values less than -2, between -2 and 2, and greater than 2 into $f'(x)$ shows that the function is increasing when $x < -2$ and $x > 2$. It is decreasing for $-2 < x < 2$.

b) Using the first derivative test and the information in part a), we find that -2 is a local max and 2 is a local min.

c) $f''(x) = 6x$, so the function is concave up when $f'' > 0$, that is when $x > 0$. It is concave down for $x < 0$, so 0 is an inflection point.

d)



6. (10 points) Consider the function $f(x) = \frac{x}{(x+1)^2}$.

a) What is the domain of f?

b) Find the critical numbers of f.

c) Find the absolute minimum and maximum of f in the interval $[0, 2]$.

Solution. a) The domain is where the denominator is not zero, that is $x \neq -1$.

b) The critical numbers are the numbers in the domain where the derivative is 0 or it does not exist:

$$f'(x) = \frac{(x+1)^2 - 2x(x+1)}{(x+1)^4} = \frac{(x+1) - 2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}.$$

So $x = 1$ is the only critical point ($x = -1$ is not a critical point, not being in the domain).

c) We compare the values of f at the critical point and at the endpoints. Since $f(0) = 0, f(1) = 1/4 = 0.25, f(2) = 2/9 = 0.22$, the absolute max is 0.25 at $x = 1$ and the absolute min is 0 at $x = 0$.