College of the Holy Cross, Fall 2008
Math 131 - Practice Exam 3 Solutions
Problems I h), VI-VIII

I h) $y' = -ex^{e-1}$. The first two terms in the numerator are constants, hence their derivative is 0.

VI. Water is leaking out of a conical tank at a rate of 1 m^3/min. The tank has height 6m and the radius of the top is 2m. Initially the tank is full. How fast is the water level dropping when the height of the water in the tank is 2m? (The volume of a cone of height h and base circle of radius r is $\frac{1}{3}\pi r^2 h$.)

Solution: First draw a picture, and call h the height of the water in the tank, and r the top radius of the water surface in the tank. The volume of water is

$$V = \frac{1}{3}\pi r^2 h,$$

and we know $dV/dt = -1$ (the minus sign is because the volume of water is decreasing). We want to find dh/dt when $h = 2$.

From similar triangles, we find:

$$\frac{r}{h} = \frac{2}{6} \quad \text{so} \quad r = \frac{h}{3},$$

and the volume in terms of h alone is $V = \frac{\pi}{27}h^3$. Taking the derivative with respect to t of this equation gives:

$$\frac{dV}{dt} = \frac{\pi}{27}3h^2\frac{dh}{dt}.$$

We can finally plug in $dV/dt = -1$, $h = 2$, and find that $dh/dt = -9/(4\pi)$. Therefore the height of the water is decreasing at $9/(4\pi)$ meters per minute.

VII. a) Find the linear approximation of the function $f(x) = \ln(x)$ at $x = 1$.

b) Using the linear approximation in part a), estimate $\ln(1.1)$.

c) Is your estimate larger or smaller than the actual value? Justify your answer.

Solution. a) We have $f'(x) = 1/x$, so $f'(1) = 1$. Also $\ln(1) = 0$ so the linear approximation of $f(x)$ at 1 is equal to:

$$L(x) = 0 + 1 \cdot (x - 1) = x - 1.$$

b) We know that $f(x)$ is well approximated by $L(x)$ for x close to 1. Taking $x = 1.1$ we find

$$\ln(1.1) \simeq L(1.1) = 1.1 - 1 = 0.1 \ .$$

c) The graph of $\ln(x)$ lies below the tangent line to the graph at $x = 1$ (draw a picture to convince yourself!). Since $L(x)$ represents the y coordinate on the tangent line, it is larger than the y coordinate on the graph, namely $f(x)$. So the approximation in part b) is larger than the actual value.

Indeed, using a calculator we find $\ln(1.1) = 0.0953....$

VIII. Find the equation of the tangent line to the curve $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$.

Solution: To find the equation of the tangent line we need its slope, that is the derivative y' at the point $(2, 1)$. Viewing y as an implicit function of x and taking the derivative of the given equation we get:

$$2x + 4xy' + 4y + 2yy' = 0$$

Solving for y' we find $y' = -\dfrac{x + 2y}{2x + y}$. Plugging in $x = 2$ and $y = 1$, we finally find the slope $y' = -4/5$. Therefore the equation of the tangent line at the point $(2, 1)$ can be found from the point-slope formula:

$$y - 1 = -\frac{4}{5}(x - 2).$$